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2-STRATIFOLDS WITH FINITE FUNDAMENTAL GROUP OR ABELIAN FUNDAMENTAL GROUP

By

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ABSTRACT

This dissertation is focused on the study of spaces called 2-stratifolds. These spaces are locally modeled on a 2-dimensional space where n-sheets meet. Unlike 2-manifolds, 2-stratifolds are not determined by their fundamental group and a complete list of 2-stratifold groups is unknown. To further understand these groups, we determine which finite groups and which abelian groups are the fundamental group of a 2-stratifold. A powerful tool for studying 2-stratifolds is the associated labelled bicolored graph. This graph essentially determines the homeomophism type. A classification of all labelled graphs that represent 1-connected trivalent 2-stratifold had been previously obtained by Gómez-Larrañaga, González-Acuña, and Heil in [7]. We extend this classification to labelled graphs that represent trivalent 2-stratifold with finite fundamental group or abelian fundamental group.

CHAPTER 1

INTRODUCTION

A closed 2-stratifold is a compact connected 2-dimensional cell complex X that can be constructed from a disjoint union of compact connected 2-manifolds W^2 and disjoint union $X^{(1)}$ of circles by attaching each component of ∂W^2 to $X^{(1)}$ via a covering map $\psi : \partial W^2 \to X^{(1)}$ with $\psi^{-1}(x) > 2$ for $x \in X^{(1)}$. Figure 1.1 is an example of this construction. These 2-stratifolds are a generalization of closed 2-manifolds, as 1-dimensional cell complexes are a generalization of 1-manifolds.

In [3], Eto, Matsuzaki, and Ozawa study the embeddability of 2-dimensional cell complexes into $\mathbb{R}^3$ and they introduce multibranched surfaces. Multibranched surfaces are a slightly more general class of 2-dimensional stratifed spaces than 2-stratifolds. A multibranched surface is constructed as a 2-stratifold except the attaching map $\psi : \partial W' \to X^{(1)}$ is from a subcollection $\partial W'$ of the components of ∂W^2. In [15], Matsuzaki and Ozawa show that all multibranched surfaces embed in $\mathbb{R}^4$.

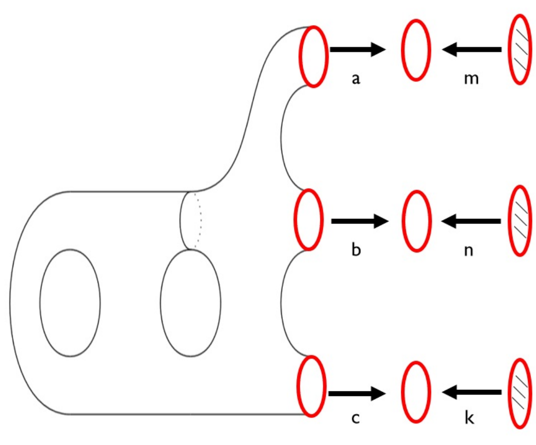


Figure 1.1: An example of a 2-stratifold.

Multibranched surfaces and 2-stratifolds where each point of $X^{(1)}$ has a neighborhood where three sheets meet are called tribranched surfaces and trivalent 2-stratifolds respectively. Tribranched surfaces and trivalent 2-stratifolds are a subclass of 2-foams. A 2-foam is a compact topological space

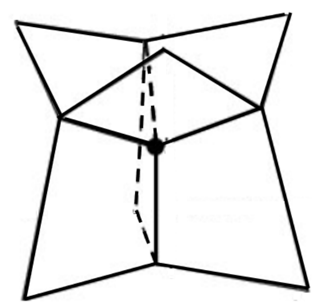


Figure 1.2: Local picture of a 2-foam.

such that each point has a neighborhood homeomorphic to a neighborhood of the 2-dimensional cell complex in figure 1.2. Reidemeister/Roseman-type moves of embedded knotted 2-foams in $\mathbb{R}^4$ have been studied in [2] by Scott Carter.

In [15], Matsuzaki and Ozawa show that multibranched surfaces are embeddable into an orientable closed 3-manifold if and only if they the set $X^{(1)}$ satisfies a regularity condition. Then Ishihara, Koda, Ozawa, and Shimokawa in [11], define a neighborhood equivalence class on embedded multibranched surfaces and give moves that determine when different embedded multibranched surfaces belong to the same class. Gómez-Larrañaga, González-Acuña, and Heil in [9] show which 2-stratifolds are spines of closed 3-manifolds. Friedl, Kitayama, Nagel show that if M is a closed 3-manifold with rank $\pi_1(M) \geq 4$ then M admits an essential tribranched surface in [4].

We know by the classification of closed 2-manifolds that closed 2-manifolds are determined uniquely by their fundamental group. From the classification we are also able to enumerate all the groups which are the fundamental group of a closed 2-manifold. In comparison, we are unable to enumerate the groups which are the fundamental group of a 2-stratifold. In fact, there are infinitely many non-homeomorphic 2-stratifolds that have isomorphic fundamental groups.

The class of groups, 2-stratifold groups, that are realizable as the fundamental group of 2-stratifolds contains many interesting finitely presented groups. For example the fundamental group of a compact 2-manifold (with or without boundary) is a 2-stratifold group. Other examples of 2-stratifold groups include free products of cyclic groups, Baumslag-Solitar groups, and F-groups. In [8], a 2-stratifold group was shown to be isomorphic to the fundamental group of a graph of groups with vertex groups that are either F-groups or cyclic groups and edge groups that are cyclic groups.

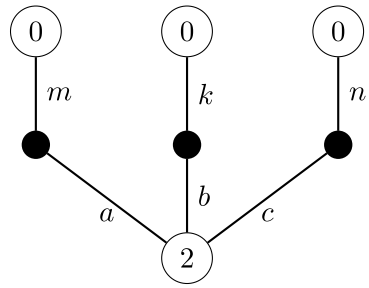


Figure 1.3: Labelled graph of the 2-stratifold in figure 1.1.

We use this graph of groups representation to determine the finite 2-stratifold groups and the abelian 2-stratifold groups. This is the main result of the chapter 2 and is given by the following theorem.

Theorem 3.4.3. *Let X be a 2-stratifold.*

1. *If X has finite fundamental group then $\pi_1(X)$ is finite cyclic, dihedral group of order $2n$, or the tetrahedral, octahedral, dodecahedral group.*
2. *If X has abelian fundamental group then $\pi_1(X)$ is either finite cyclic, dihedral group of order 4, $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, or $\mathbb{Z} \times \mathbb{Z}_n$.*

The homeomophism class of a 2-stratifold is determined by a bicoloured labelled graph. For a 2-stratifold X this bicoloured labelled graph Γ_X has white vertices that correspond to the components W^2, the black vertices that correspond to components of $X^{(1)}$, and an edge is a component of $W^2 \cap X^{(1)}$, where the label on an edge is the degree of the map $\psi : \partial W^2 \to X^{(1)}$. In [7], Gómez-Larrañaga, González-Acuña, and Heil gave a classification of bicoloured labelled graphs that represent 1-connected trivalent 2-stratifolds. Then in [10], they obtain necessary and sufficent conditions on X and on the graph Γ_X such that $\pi_1(X) = \mathbb{Z}$. They then give a classification of trivalent 2-stratifolds with fundamental group $\mathbb{Z}$. This classification is in terms of conditions that can be read off the labelled graph Γ_X.

In this dissertation, we extend the classification to trivalent 2-stratifolds with finite fundamental group or abelian fundamental group. For trivalent 2-stratifolds with finite fundamental groups we first find the trivalent 2-stratifold groups. This is given by the following theorem.

Theorem 5.2.2. *Let Γ be a bicolored trivalent graph. If X_Γ has finite fundamental group then $\pi_1(X_\Gamma)$ is isomorphic to either $\mathbb{Z}_{2^{k+1}}$, $\mathbb{Z}_{3*2^k}$, $D_{2^{k+1}}$ where $k \geq 0$.*

The classification of trivalent 2-stratifolds with finite fundamental group follows from theorem 5.2.2. This classification is in terms of conditions that can be read off the labelled graph Γ_X and is given by corollaries 5.2.3-5.2.7.

We then give a classification of trivalent 2-stratifold groups with $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}_m$. Similarly we find the trivalent 2-stratifold groups that are of the form $\mathbb{Z} \times \mathbb{Z}_m$. This is given by the following theorem.

Theorem 6.3.2. *Let Γ be a bicolored trivalent graph. If $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}_m$ for $m > 1$ then $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}_{2^k}$ for $k > 0$.*

The classification of trivalent 2-stratifolds with fundamental group $\mathbb{Z} \times \mathbb{Z}_{2^k}$ is then given by theorem 6.3.3 and classification of trivalent 2-stratifolds with fundamental group $\mathbb{Z} \times \mathbb{Z}$ is given by theorem 6.4.4.

1.1 Outline

The chapter two starts by introducing 2-stratifolds X and how to represent a 2-stratifold with a bicolored labelled graph Γ_X. We review how to compute a presentation of the fundamental group of a 2-stratifold from the graph Γ_X. Altering the graph Γ_X changes the homoemorphism type of a 2-stratifold. But certain alterations of the graph Γ_X changes the fundamental group of a 2-stratifold in a predictable way. We introduce operations that are used to determine the fundamental group of X. We then find necessary conditions on Γ for when X has finite fundamental group or abelian fundamental group.

In the third chapter, we study the fundamental groups of 2-stratifolds. The main purpose of this chapter is to determine the finite 2-stratifold groups and the abelian 2-stratifold groups. To do this we study the structure of a 2-stratifold group by representing the group as the fundamental group of a graph of groups. We show that the graph of groups corresponding to a finite 2-stratifold group collapses to a single vertex and that the graph of groups corresponding to an abelian 2-stratifold group collapses to either a single vertex or a single vertex along with a single edge.

In chapter four and chapter five we focus on determining which bicolored trivalent graphs Γ represent a trivalent 2-stratifold X_Γ with finite fundamental group. In chapter 3 we find a set of necessary conditions on Γ so that X_Γ has finite fundamental group. Then in the first part of chapter four, we find the finite fundamental groups of X_Γ where Γ satisfies the necessary conditions from chapter 3. In the second part of chapter four, we give the necessary and sufficient conditions on Γ so that X_Γ has finite fundamental group.

In chapter six we focus on determining which bicolored trivalent graphs Γ represent a trivalent 2-stratifold X_Γ with abelian fundamental group. We first determine the necessary and sufficient conditions on a graph Γ that represents a trivalent 2-stratifold X_Γ with fundamental group $\mathbb{Z} \times \mathbb{Z}_{2^k}$. Then we complete the classification of trivalent 2-stratifolds with abelian fundamental group by determining when a trivalent 2-stratifold X_Γ has fundamental group $\mathbb{Z} \times \mathbb{Z}$.

CHAPTER 2

DEFINITIONS AND PROPERTIES OF 2-STRATIFOLDS

The purpose of this chapter is to introduce the necessary definitions and theorems needed for the study and classification of trivalent 2-stratifolds.

We will begin by reviewing basic definitions regarding 2-stratifolds X and their associated labelled graph Γ_X that were introduced in [5]. This will include the presentation of the fundamental group arising from the associated graph Γ_X and operations on Γ_X that alters the fundamental group in a controlled manner. The group presentation and the operations appear in [6] and [8].

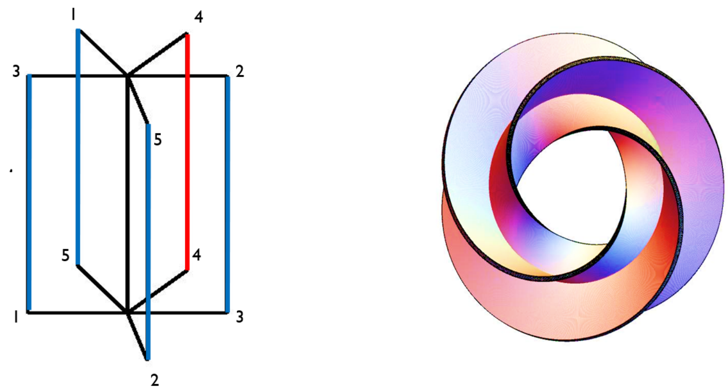
We then find necessary conditions on Γ_X for X to have either finite fundamental group or abelian fundamental group. For a 2-stratifold X with finite fundamental group these conditions are given by Lemma 2.3.3. For a 2-stratifold X with abelian fundamental group these conditions are given by Lemma 2.3.5.

2.1 Preliminaries

A **2-stratifold** X is a compact, Hausdorff space X that contains a closed (possibly disconnected) 1-manifold $X^{(1)}$ as a closed subspace with the following property: Each point $x \in X^{(1)}$ has a neighborhood homeomorphic to $\mathbb{R} \times CF$, where CF is the open cone on the finite set F with cardinality > 2, and where $X \setminus X^{(1)}$ is a (possibly disconnected) 2-manifold.

A component B of $X^{(1)}$ has a regular neighborhood denoted by $N(B) = N_\pi(B)$. The regular neighborhood $N_\pi(B)$ is homeomorphic to the mapping cylinder of f where if π is the partition $n_1 + n_2 + \ldots + n_r$ of d, the map $f : \tilde{B} \to B$ is from a closed 1-manifold with components $\tilde{B}_1$, $\tilde{B}_2$,$\ldots, \tilde{B}_r$ and the restriction of f to $\tilde{B}_i$ is an n_i-fold covering $1 \leq i \leq r$. The space $N_\pi(B)$ is determined by the partition of d.

For a 2-stratifold X there is an associated bipartite graph Γ_X embedded in X. For disjoint components B and B' of $X^{(1)}$ allow $N(B)$ and $N(B')$ be chosen sufficiently small so that $N(B)$ and $N(B')$ are disjoint. The white vertices w_i of the graph Γ_X are the components W_i of $M =$



(a.) (b.)

Figure 2.1: Regular neighborhoods $N(B)$ determined by the partitions $4+1$ and 5 of 5.

$\overline{X \setminus \cup_i N(B_i)}$ for all components B_i of $X^{(1)}$. The black vertices b_i of graph Γ_X correspond to the regular neighborhood $N(B_i)$. An edge is e_{ij} is component of E_{ij} of ∂M that joins b_j and w_i if $W_j \cap N(B_i) = E_{ij}$.

We label the white vertices w_i of graph Γ_X with the genus of the corresponding surface W_i. By convention, we assign a negative genus g to a nonorientable surface. Each edge of Γ_X is labeled by an integer k, where k is the summand of the partition π corresponding to the boundary component E of $N(B_i)$.

A presentation of the fundamental group $\pi_1(X)$ arises from the graph Γ_X. For a given white vertex w, the corresponding compact 2-manifold W has oriented boundary curves $c_1, \ldots, c_p$ with fundamental group

$$\pi_1(W) = \{c_1, \ldots, c_p, y_1, \ldots, y_n : c_1 \ldots c_p q = 1\},$$

where if W is orientable and genus $g = 2n$ then $q = [y_1, y_2] \ldots [y_{2g-1}, y_{2g}]$ and if W is nonorientable and genus $g = -n$ then $q = y_1^2 \ldots y_n^2$.

Let $\mathcal{B}$ be the set of black vertices, $\mathcal{W}$ the set of white vertices and choose a fixed maximal tree T of Γ_X. We choose orientations of the black vertices and of all boundary components of M such that all labels of edges in T are positive. Then $\pi_1(X_\Gamma)$ has the presentation with generators: $\{b\}_{b\in\mathcal{B}}$

$\{c_1,\ldots,c_p,y_1,\ldots,y_n\}$, one set for each $w\in\mathcal{W}$

$\{t_i\}$, one t_i for each edge in $\Gamma_X\setminus T$ between w and b

relations: $c_1\ldots c_pq=1$ one for each $w\in\mathcal{W}$

$b^m=c_i$ for each edge $e_i\in T$ between w and b with edge label $m\geq 1$

$t^{-1}c_it=b^{\pm m}$ one for each edge e in $\Gamma_X\setminus T$ between w and b with edge label $m\geq 1$

Notation 2.1.1. *The labelled bipartite graph associated to a 2-stratifold X is denoted by Γ_X and X is denoted by X_Γ. If Γ is a bipartite labelled tree then there is a unique 2-stratifold X such that $\Gamma_X=\Gamma$.*

2.2 Operations on 2-stratifold graphs

For a given bipartite labelled graph Γ_X there are operations on Γ_X that produce a bipartite labelled graph Γ' such that there is an epimorphism (or isomorphism) from $\pi_1(X_\Gamma)$ to $\pi_1(X_{\Gamma'})$. We review these operations here.

The following was shown in [6].

Lemma 2.2.1. *There is a retraction $r:X\to\Gamma_X$ such that $r^{-1}(b)$ is a singular curve $B\in X^{(1)}$ and $r^{-1}(w)$ is a 2-manifold W.*

Let Γ_0 be a subgraph of Γ_X and let $Y=r^{-1}(\Gamma_0)$. The space Y contains boundary curves corresponding to $St(\Gamma_0)-\Gamma_0$, where $St(\Gamma_0)$ is the closed star of Γ_0 in Γ_X. Denote the labelled edges of $St(\Gamma_0)-\Gamma_0$ adjacent to a black vertex of Γ_X as E. Attach disks with a degree 1 attaching maps to the boundary curves of Y. The resulting space is a 2-stratifold $Y'=X_{\Gamma'}$ where Γ' is obtained by deleting the complement of $\Gamma_0\cup E$ from Γ_X then attaching white vertices of genus zero to the labelled edges of E. We say Γ' is obtained from Γ by pruning at Γ_0.

In Fig.1.2, the graph Γ_0 is the linear graph $w_0-b_1-w_1-b_2-w_2$ where the white vertices are of genus g,g_0,g_4 and the edge labels are m_1,n_1,m_2,n_2.

Remark 2.2.2. *If Γ' is obtained from Γ by pruning at Γ_0, then there is an epimorphism from $\pi_1(X_\Gamma)$ to $\pi_1(X_{\Gamma'})$.*

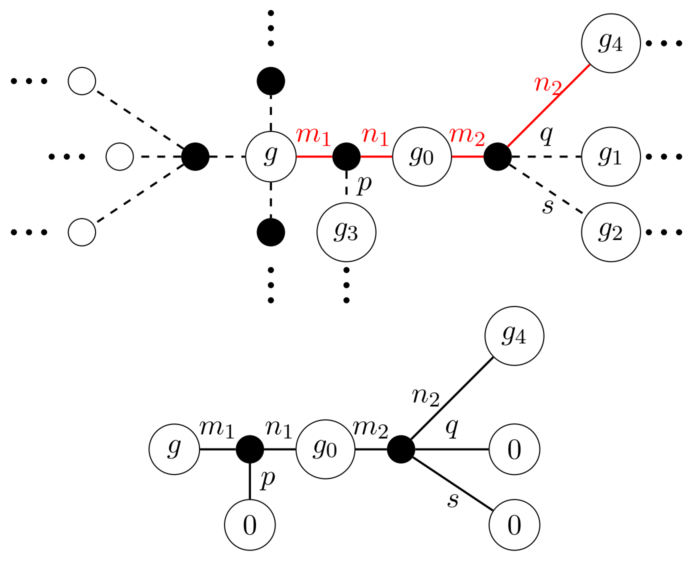


Figure 2.2: Pruning Γ_X at Γ_0 results in Γ'

A linear bipartite labelled graph L with successive vertices $w_0 - b_1 - w_1 - \ldots - b_r - w_r$, successive labels $m_1, n_1, \ldots, m_r, n_r$ where m_i (resp. n_i) is the label of the edge joining b_i to w_{i-1} (resp. w_i) for $r = 1, \ldots, r$ will be denoted by $L = L(m_1, n_1, \ldots, m_r, n_r)$. A linear bipartite labelled graph L with successive vertices $b_1 - w_1 - \ldots - b_r - w_r$ and successive labels $n_1, \ldots, m_r, n_r$ will be denoted by $L = L(n_1, \ldots, m_r, n_r)$. A linear subgraph $L(m_1, n_1, \ldots, m_r, n_r)$ of Γ_X (resp. $L(n_1, \ldots, m_r, n_r)$) will be called **terminal** if w_r is a terminal vertex of Γ and vertices b_i, w_i for $i > 0$ (resp. b_{i+1}, w_i for $i > 0$) are of degree < 3.

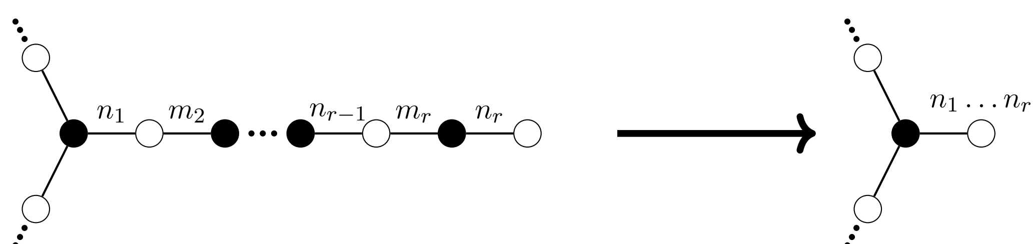


Figure 2.3: L-pruning $L(n_1, \ldots, m_r, n_r)$ from Γ_X

Let $L = L(m_1, n_1, \ldots, m_r, n_r)$ be a terminal linear subgraph of Γ where the initial vertex w_0 has genus g and all other white vertices in L have genus 0. Let $L(1, n_1 \ldots n_r)$ be a linear graph whose initial vertex has genus g while all other vertices have genus 0. **L-pruning Γ at** $L(m_1, n_1, \ldots, m_r, n_r)$ is the process of replacing $L(m_1, n_1, \ldots, m_r, n_r)$ with $L(1, n_1 \ldots n_r)$. Let $L(n_1, \ldots, m_r, n_r)$ be a terminal linear subgraph of Γ where all white vertices in L have genus 0. The process of replacing $L(n_1, \ldots, m_r, n_r)$ with $L(n_1 \ldots n_r)$ will also be called **L-pruning Γ at** $L(n_1, \ldots, m_r, n_r)$.

Lemma 2.2.3. *Let X be a 2-stratifold.*

1. *Let $L = L(m_1, n_1, \ldots, m_r, n_r)$ be a terminal linear subgraph of Γ_X where the initial vertex w_0 has genus g and all other white vertices in L have genus 0. Let Γ' be obtained from Γ_X by L-pruning Γ_X at $L(m_1, n_1, \ldots, m_r, n_r)$.*

   If $gcd(m_i, n_j) = 1$ for $1 \leq i \leq j \leq r$ then $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$.

2. *Let $L(n_1, \ldots, m_r, n_r)$ be a terminal linear subgraph of Γ_X where all white vertices in L have genus 0. Let Γ' be obtained from Γ_X by L-pruning Γ_X at $L(n_1, \ldots, m_r, n_r)$.*

   If $gcd(m_i, n_j) = 1$ for $1 < i \leq j \leq r$ then $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$.

Proof. (1.) This was shown in [6].

(2.) Let the terminal linear subgraph $L(n_1, \ldots, m_r, n_r)$ of Γ_X have vertices ordered as $b_1 - w_1 - b_2 - ... - b_r - w_r$ where w_r is a terminal vertex of Γ_X and successive edge labels are $n_1 - \ldots - m_r - n_r$. Let S be the subgraph of $L(n_1, \ldots, m_r, n_r)$ with initial vertex w_1 and terminal vertex w_r. L-prune the graph Γ_X at S. In the resulting graph Γ'', the terminal linear graph $L(n_1, \ldots, m_r, n_r)$ of Γ_X has been replaced by the terminal linear subgraph $L(n_1, 1, n_2 \ldots n_r)$. Let the terminal linear subgraph $L(n_1, 1, n_2 \ldots n_r)$ of Γ'' have vertices order as $b_1 - w_1'' - b_2'' - w_3''$ where w_3'' is the terminal vertex of Γ''. The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma''})$ by part (1.) of this Lemma.

Let Γ' be obtained from Γ_X by L-pruning the graph Γ_X at $L(n_1, \ldots, m_r, n_r)$. Let the terminal linear subgraph $L(n_1 n_2 \ldots n_r)$ of Γ' have vertices $b_1 - w_1'$ where w_1' is a terminal vertex of Γ'.

Let G be obtained from Γ_X by deleting the graph $-w_1 - b_2 - ... - b_r - w_r$. Let the generators of $\pi_1(X_G)$ be denoted $\mathcal{G}$ and the relations of $\pi_1(X_G)$ be denoted $\mathcal{R}$.

If b_1, b_2 are generators of $\pi_1(X_{\Gamma''})$ corresponding to the curves $r^{-1}(b_1)$ and $r^{-1}(b_2'')$ for vertices b_1 and b_2'' in Γ'' then the presentation of $\pi_1(X_{\Gamma''})$ is

$$\{\mathcal{G}, b_2 | \mathcal{R},\, b_1^{n_1} = b_2,\, b_2^{n_2 \ldots n_r} = 1\}.$$

Removing the generator b_2 from the presentation of $\pi_1(X_{\Gamma''})$ results in

$$\{\mathcal{G} | \mathcal{R},\, b_1^{n_1 n_2 \ldots n_r} = 1\}.$$

This presentation is equivalent to the presentation of $\pi_1(X_{\Gamma'})$. Then $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$.

□

Let b be black vertex of Γ_X that is the initial vertex of $k > 1$ terminal linear branches $L_i(n_i)$. For all i, let $L_i(n_i)$ have a white vertex of genus 0. **Reducing the degree of** b is the process of replacing the terminal linear branches $L_i(n_i)$ of Γ_X with a single terminal linear branch $L(n')$.

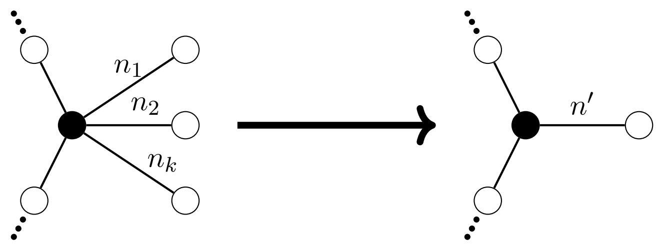


Figure 2.4: Reducing degree of a black vertex.

Lemma 2.2.4. *Let X be a 2-stratifold whose graph Γ_X contains black vertex b that is the initial vertex of $k > 1$ terminal linear branches $L_i(n_i)$ such that the white vertex of $L_i(n_i)$ has genus* 0. *Let Γ' be obtained from Γ_X by reducing the degree of b.*

If $n' = gcd(n_1, \ldots, n_k)$ then $\pi_1(X) \cong \pi_1(X_{\Gamma'})$.

Proof. Let G be obtained from Γ_X by deleting the $\cup_i(L_i \setminus b)$ for all $1 \leq i \leq k$. Let the generators of $\pi_1(X_G)$ be denoted $\mathcal{G}$ and the relations of $\pi_1(X_G)$ be denoted $\mathcal{R}$ and let b be the generator of $\pi_1(X_\Gamma)$ corresponding to the curve $r^{-1}(b)$.

Suppose that $k = 2$. Then the presentation of $\pi_1(X_\Gamma)$ is

$$\{\mathcal{G}|\mathcal{R},\, b^{n_1},\, b^{n_2}\}.$$

This presentation is equivalent to

$$\{\mathcal{G}|\mathcal{R},\, b^m\}$$

where $m = gcd(n_1, n_2)$. Then $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$.

Suppose that $k > 2$. Then the presentation of $\pi_1(X_\Gamma)$ is

$$\{\mathcal{G}|\mathcal{R},\, b^{n_1}, \dots, b^{n_k}\}.$$

The presentation is equivalent to

$$\{\mathcal{G}|\mathcal{R},\, b^{n_1}, b^{m}\}.$$

where $m = gcd(n_2 \dots, n_k)$. This presentation is equal to

$$\{\mathcal{G}|\mathcal{R},\, b^{m'}\},$$

where $m' = gcd(n_1, m)$. Since gcd is associative we have $m' = gcd(n_1, \dots, n_r)$. Then $\pi_1(X) \cong \pi_1(X_{\Gamma'})$

□

2.3 Graphs of 2-stratifolds with finite or abelian fundamental group

In this section we find some necessary conditions on Γ_X for $\pi_1(X)$ to be either a finite group or an abelian group. The following lemma was shown in [10]. We denote the closed surface of genus g to be S_g. Note that since a black vertex of Γ corresponds to a singular curve of X, a terminal edge of Γ incident to a black vertex has label ≥ 3.

Lemma 2.3.1. *Let X be 2-stratifold with graph Γ_X.*

1. *If Γ_X has two black terminal vertices with incident edge labels $m, n \geq 3$, then there is an epimorphism $\pi_1(X) \to \mathbb{Z}_m \star \mathbb{Z}_n$.*
2. *If Γ_X has a black terminal vertex with with incident edge label $m \geq 3$ and contains a white vertex of genus g then then there is an epimorphism $\pi_1(X) \to \mathbb{Z}_m \star \pi_1(S_g)$.*
3. *If Γ_X contains two white vertices of genera g_1, g_2 then then there is an epimorphism $\pi_1(X) \to \pi_1(S_{g_1}) \star \pi_1(S_{g_2})$.*

Lemma 2.3.2. *Let X be 2-stratifold with graph Γ_X.*

1. *If $\pi_1(X)$ is finite then Γ_X is a tree.*
2. *If $\pi_1(X)$ is abelian then Γ_X is a tree or homotopy equivalent to S^1.*

Proof. The retraction $r : X \to \Gamma_X$ induces an epimorphism $r_\star : \pi_1(X) \to \pi_1(\Gamma_X)$. $\square$

From these lemmas we can conclude the following.

Lemma 2.3.3. *Let X be 2-stratifold with graph Γ_X. If $\pi_1(X)$ is finite then Γ_X is a tree that satifies one of the following set of conditions:*

1. *Γ_X has all white vertices of genus 0, one black terminal vertex and all other terminal vertices are white.*
2. *Γ_X has at most one white vertex of genus -1 while all other white vertices are genus 0, and all terminal vertices are white.*

Proof. By Lemma 2.3.2, Γ_X is a tree. If w is a white vertex of Γ_X then pruning Γ_X at w results in a closed 2-manifold W' with finite fundamental group. The 2-manifold W' is either a 2-sphere or real projective plane.

By Lemma 2.3.1, Γ_X contains at most one white vertex of genus -1 or one black terminal vertex.

If Γ_X contains one black terminal vertex then all other terminal vertices are white and all white vertices are genus zero by Lemma 2.3.1.

If Γ_X contains a white vertex of genus -1 then all other white vertices are genus zero and all terminal vertices are white by Lemma 2.3.1. $\square$

The following lemma was also shown in [10].

Lemma 2.3.4. *Let X be 2-stratifold with graph Γ_X that is homotopy equivalent to S^1.*

1. *If Γ_X has a black terminal vertex then there is an epimorphism $\pi_1(X) \to \mathbb{Z} \star \mathbb{Z}_n$ for some $n \geq 3$.*
2. *If Γ_X contains a white vertex of genus g then then there is an epimorphism $\pi_1(X) \to \mathbb{Z} \star \pi_1(S_g)$.*

For a 2-stratifold X with abelian fundamental group we can conclude the following necessary conditions on Γ_X.

Lemma 2.3.5. *Let X be 2-stratifold with graph Γ_X. If $\pi_1(X)$ is abelian then Γ_X is satifies one of the following set of conditions:*

1. *Γ_X is homotopy equivalent to S^1, all white vertices are genus 0, and all terminal vertices are white.*

2. Γ_X *is a tree, all white vertices of genus* 0*, and contains at most one black terminal vertex while all other terminal vertices are white.*

3. Γ_X *is a tree, all terminal vertices are white, and all but at most one white vertex is of genus 0. At most one white vertex is of genus* 1 *or* -1*.*

Proof. By Lemma 2.3.2, Γ_X is a tree or homotopy equivalent to S^1.

By Lemma 2.3.4, If Γ_X is homotopy equivalent to S^1 then all white vertices are genus 0 and all terminal vertices are white.

Assume that Γ_X is a tree. If w is a white vertex of Γ_X then pruning Γ_X at w results in a closed 2-manifold W' with abelian fundamental group. The 2-manifold W' is either a 2-sphere, real projective plane, or a torus. By Lemma 2.3.1, Γ_X contains at most one black terminal vertex or one white vertex of genus $g = \pm 1$.

If Γ_X contains one black terminal vertex then all other terminal vertices are white and all white vertices are genus zero by Lemma 2.3.1.

If Γ_X contains a white vertex of genus $g \pm 1$ then all other white vertices are genus zero and all terminal vertices are white by Lemma 2.3.1. □

CHAPTER 3

2-STRATIFOLD GROUPS

A group G is called a **2-stratifold group** if $G = \pi_1(X)$ for a 2-stratifold X. The goal of this chapter is to determine what finite groups and what abelian groups are 2-stratifold groups. All groups in this chapter are assumed to be finitely presented.

We first introduce the graph of groups and analyze graph of groups whose fundamental group is finite or abelian. Then we will represent $\pi_1(X)$ as the fundamental group of a graph of groups, as in [8], to show the following.

Theorem 3.4.3. *Let X be a 2-stratifold.*

1. *If X has finite fundamental group then $\pi_1(X)$ is finite cyclic, dihedral group of order $2n$, or the tetrahedral, octahedral, dodecahedral groups.*
2. *If X has abelian fundamental group then $\pi_1(X)$ is either finite cyclic, dihedral group of order 4, $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, or $\mathbb{Z} \times \mathbb{Z}_n$.*

3.1 Graph of groups

We recall some related terminology and properties of graph of groups. We then determine the reduced graph of groups for graph of groups whose fundamental group is either finite or abelian.

An **abstract graph** Y consists of two sets: $V = V(Y)$, vertices, and $E = E(Y)$, (oriented) edges, together with maps $E \to V \times V$, $e \to (o(e), t(e))$ (the originating and terminal vertices of e), and $E \to E$, $e \to \bar{e}$ (reversal of orientation) such that $e = \bar{\bar{e}}$, $e \neq \bar{e}$, $t(e) = o(\bar{e})$, and $o(e) = t(\bar{e})$. A **graph of groups** (G, Y) consists of an abstract graph Y, two families of groups $\{G_v | v \in V(Y)\}$, $\{G_e | e \in E(Y)\}$ such that $G_e = G_{\bar{e}}$, and a family of monomorphisms $\{f_e\}$ with $f_e : G_e \to G_{t(e)}$, $f_{\bar{e}} : G_{\bar{e}} \to G_{o(e)}$.

For a graph of groups (G, Y), the group $F(G, Y)$ is generated by the vertex groups G_v and elements e corresponding to the elements of $E(Y)$, subject to the relations $\bar{e} = e^{-1}$ and $ef_e(x)e^{-1} = f_{\bar{e}}(x)$ for all $x \in G_e$ and for each $e \in E(Y)$. For a fixed vertex v_0, the **fundamental group** $\pi_1(G, Y, v_0)$ **of**

the graph of groups (G, Y) is the subgroup of $F(G, Y)$ generated by all words

$$w = r_0 e_1 r_1 e_2 \ldots e_n r_n$$

where $v_0 - v_1 - v_2 - \ldots - v_n$ is an edge path with initial and terminal vertex $v_0 = v_n$ (i.e. a cycle based at v_0), successive edges e_i (joining v_{i-1} to v_i) and $r_i \in G_{v_i}$. The word $w = r_0 e_1 \ldots e_n r_n$ of length n is **reduced**, if for $n = 0$, $r_0 \neq 1$; for $n \geq 1$, $r_i \notin f_e(G_{e_i})$, for each index i such that $e_{i+1} = \bar{e}_i$. The group $\pi_1(G, Y, v_0)$ is independent of the choice of v_0.

Serre showed the following in [17]

Lemma 3.1.1. *If $w \in \pi_1(G, Y, v_0)$ is a reduced word then $w \neq 1$ in $\pi_1(G, Y, v_0)$. If (G, Y) is a graph of groups, the homomorphism $G_v \to \pi_1(G, Y, v_0)$ is injective.*

A **subgraph of subgroups** (G', Y') of (G, Y) is a graph of groups where Y' is a connected subgraph of Y, $G'_v \leq G_v$ for all v in Y', and for all $e \in E(Y')$, $G'_e \leq G_e$ and $f'_e = f_e|_{G'_e}$.

Bass proved the next lemma in [1] (pgs. 10, 24).

Lemma 3.1.2. *If (G', Y') is a subgraph of groups of (G, Y), then the natural homomorphism $i_\star : \pi_1(G', Y', v_0) \to \pi_1(G', Y', v_0)$ is injective.*

For a graph of groups (G, Y) where Y contains one edge $\{e, \bar{e}\}$ the fundamental group $\pi_1(G, Y, v_0)$ is called a free product with amalgamation, denoted $G_{v_0} \star_{G_e} G_{v_1}$, if Y contains two vertices v_0, v_1 and an HNN group, denoted $G_{v_0} \star_{G_e}$, if Y contains a single vertex v_0. The HNN group, $G_{v_0} \star_{G_e}$, will also be referred to as an HNN extension of G_{v_0} along G_e.

Corollary 3.1.3. *Let (G, Y) be a graph of groups where $G = \pi_1(G, Y, v_0)$. Let Y contain one edge $\{e, \bar{e}\}$.*

1. *If $o(e) \neq t(e)$, $f_e, f_{\bar{e}}$ are not surjective, and $G_{o(e)}$, $G_{t(e)}$ are nontrivial then G is not finite and not abelian.*

2. *Let $o(e) = t(e)$ and $G_{o(e)}$ are nontrivial. If $f_e(G_e), f_{\bar{e}}(G_e)$ are proper subgroups of $G_{o(e)}$ then G is not abelian.*

Proof. We write $f_e, f_{\bar{e}}$ as inclusions so that $G_e < G_{v_1}$, $G_{\bar{e}} < G_{v_0}$.

(1.) Let $v_0 = o(e)$ and $v_1 = t(e)$. Let (H, X) be a subgraph of subgroups (G, Y) where $H_v = G_v$ for all $v \in V(X)$, $H_e = G_e$ for all $e \in E(X)$, and X consists of two vertices v_0, v_1 and a single edge $\{e, \bar{e}\}$. The fundamental group $\pi_1(H, X, v_0) = N$ is a subgroup of G. The group N is the free product with amalgamation $G_{v_0} \star_{G_e} G_{v_1}$. There exists $a \in G_{v_0}$ and $b \in G_{v_1}$ such that $a \notin G_{\bar{e}}$ and $b \notin G_e$. The word $(ab)^k$ is a reduced word in N for all k and by lemma 3.1.1 $(ab)^k \neq 1$ in N. The word ab has infinite order. The word $aba^{-1}b^{-1}$ is a reduced word in N and $aba^{-1}b^{-1} \neq 1$ in N.

(2.) Let $v_0 = o(e)$ and let (H, X) be a subgraph of subgroups (G, Y) where $H_v = G_v$ for all $v \in V(X)$, $H_e = G_e$ for all $e \in E(X)$, and X consists of a single vertex v_0 and a single edge $\{e, \bar{e}\}$. The fundamental group $\pi_1(H, X, v_0) = N$ is a subgroup of G. The group N is the HNN group $G_{v_0} \star_{G_e}$. If $G_e \cup G_{\bar{e}} \neq G_{v_0}$ there exists $a \in G_{v_0}$ such that $a \notin G_e$ and $a \notin G_{\bar{e}}$. The word $ata^{-1}t^{-1}$ is a reduced word in N. Suppose that $G_e \cup G_{\bar{e}} = G_{v_0}$. Let $a \in G_e$ such that $a \notin G_e \cap G_{\bar{e}}$. Then $a \notin G_{\bar{e}}$. (We can find such an a since if $G_e \cap G_{\bar{e}} = G_{v_0}$ then $G_e = G_{v_0}$ but G_e,$G_{\bar{e}}$ are proper subgroups.) We claim that $ata^{-1}t^{-1}$ is a reduced word. If $ata^{-1}t^{-1} = 1$ then for the subword $ta^{-1}t^{-1}$ the element a^{-1} is contained in $G_{\bar{e}}$. If $a^{-1} \in G_{\bar{e}}$ then $a \in G_{\bar{e}}$. But $a \notin G_{\bar{e}}$. Therefore $ata^{-1}t^{-1}$ is a reduced word.

□

Remark 3.1.4. *A necessary requirement for the fundamental group $\pi_1(G, Y, v_0)$ of (G, Y) where Y is a vertex v, an edge e, and $o(e) = t(e)$ to be abelian is that at least one of $f_e(G_e), f_{\bar{e}}(G_e)$ is not a proper subgroup. The images $f_e(G_e)$ and $f_{\bar{e}}(G_e)$ are isomorphic to each other (as groups). Hence if $\pi_1(G, Y, v_0)$ is abelian then $f_e(G_e)$ and $f_{\bar{e}}(G_e)$ are isomorphic to G_v. Corollary 3.1.3 does not imply the stronger condition that at least one of the maps f_e, $f_{\bar{e}}$ are isomorphisms. But this follows from a similiar proof to (2.).*

An edge e of a graph of groups (G, Y) is said to be **trivial** if $o(e) \neq t(e)$ and f_e is an isomorphism. An edge e of a graph of groups (G, Y) where $G_{t(e)} = \{\emptyset\}$ and $o(e) \neq t(e)$ is trivial by this definition. **Collapsing a trivial edge** e of a graph of groups (G, Y) is the process constructing a new graph of groups (G', Y') where Y' is obtained from Y by contracting $\{e, \bar{e}\}$ to a point E, set $G_E := G_{o(e)}$, and $G' = G$ on all remaining edges and vertices. The fundamental group of (G', Y') is isomorphic to the fundamental group of (G, Y). A graph of groups with no trivial edge is said to be **reduced**.

Let Y be an abstract graph. The realization of Y is the topological graph $\mathbf{Y}$ with vertices $v(Y)$ and edges corresponding to the edges $\{e, \bar{e}\}$.

Lemma 3.1.5. *Let (G, Y) be a graph of groups with a finite graph Y.*

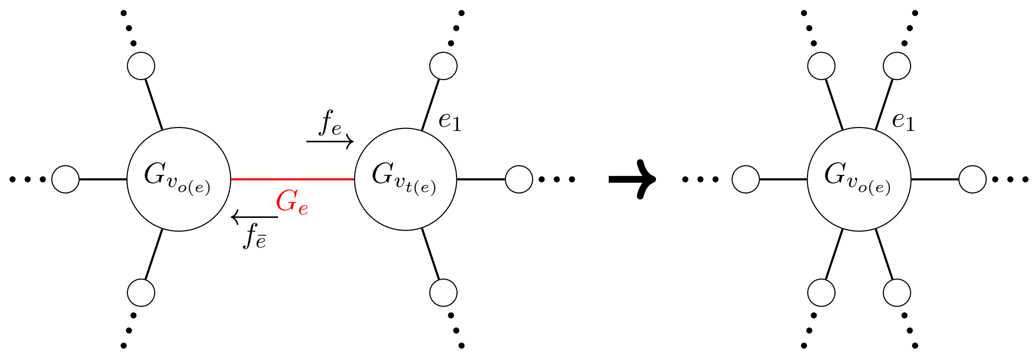


Figure 3.1: Collapsing a trivial edge.

1. *If (G, Y) is a graph of groups where $\pi_1(G, Y, v_0)$ is finite then $\pi_1(G, Y, v_0) \cong \pi_1(G', Y', v_0')$ such that (G', Y') is a reduced graph of groups where the graph Y' is a vertex v_0' with no edges and the vertex group $G_{v_0'}$ of (G', Y') is isomorphic to a vertex group G_w of (G, Y).*

2. *If (G, Y) is a graph of groups where $\pi_1(G, Y, v_0)$ is abelian then $\pi_1(G, Y, v_0) \cong \pi_1(G', Y', v_0')$ such that (G', Y') is a reduced graph of groups where the graph Y' is vertex v_0' with no edges or a vertex v_0' with a single edge e and the vertex group $G_{v_0'}$ of (G', Y') is isomorphic to a vertex group G_w of (G, Y).*

Proof. Let $\mathbf{Y}$ be the realization of Y. For any graph of groups (G, Y) there is a surjective homomorphism $\pi_1(G, Y, v_0) \to \pi_1(\mathbf{Y}, v_0)$ where $\pi_1(\mathbf{Y}, v_0)$ is the fundamental group of the graph $\mathbf{Y}$. If (G, Y) is a graph of groups where $\pi_1(G, Y, v_0)$ is finite then $\mathbf{Y}$ is a tree. If (G, Y) is a graph of groups where $\pi_1(G, Y, v_0)$ is abelian then $\mathbf{Y}$ is a tree or homotopy equivalent to S^1.

(1.) For a graph of groups (G, Y) where the graph Y contains a single vertex, the graph Y must contain no edges by the previous paragraph.

Otherwise, by induction, we assume that for a graph of groups (G, Y) where $\pi_1(G, Y, v_0)$ is finite and Y contains $n - 1$ vertices then $\pi_1(G, Y, v_0) \cong \pi_1(G', Y', v_0')$ where (G', Y') is a reduced graph of groups such that Y' is a vertex v_0' and the vertex group $G_{v_0'}$ of (G', Y') is isomorphic to a vertex group G_w of (G, Y).

Suppose that (G, Y) is a graph of groups where $\pi_1(G, Y, v_0)$ is finite and Y contains n vertices. Let (H, X) be a subgraph of subgroups (G, Y) where $H_v = G_v$ for all $v \in V(X)$, $H_e = G_e$ for all $e \in E(X)$, and X consists of two vertices v_1, v_2 and a single edge $\{e\}$ incident to v_1, v_2. Let $v_1 = o(e)$ and $v_2 = t(e)$. If $\{e, \bar{e}\}$ are nontrivial edges in (G, Y), then the fundamental group $\pi_1(H, X, v_1)$ is

$G_{v_1} \star_{G_e} G_{v_2}$, which is infinite by corollary 3.1.3. But $\pi_1(H, X, v_1)$ is a subgroup of $\pi_1(G, Y, v_1)$ and every subgroup of a finite group is finite. At least one edge e' of $\{e, \bar{e}\}$ is trivial in (G, Y). Let (G', Y') be the graph of groups obtained by collapsing the trivial edge e' of the graph of groups (G, Y). In (G', Y'), Y' contains $n-1$ vertices.

(2.) For (G, Y) where $\pi_1(G, Y, v_0)$ is abelian, we proceed similarly. For a graph Y with a single vertex, the graph Y contains either no edges or one edge by the initial paragraph.

We assume that if Y contains $n-1$ vertices then $\pi_1(G, Y, v_0) \cong \pi_1(G', Y', v_0')$ such that (G', Y') is a reduced graph of groups where the graph Y' is v_0' vertex with no edges or a vertex v_0' with a single edge and the vertex group $G_{v_0'}$ of (G', Y') is isomorphic to a vertex group G_w of (G, Y).

Suppose that (G, Y) is a graph of groups where $\pi_1(G, Y, v_0)$ is abelian and Y contains n vertices. Let (H, X) be a subgraph of subgroups (G, Y) where $H_v = G_v$ for all $v \in V(X)$, $H_e = G_e$ for all $e \in e(X)$, and X consists of two vertices v_1, v_2 and a single edge e incident to v_1, v_2. Let $v_1 = o(e)$ and $v_2 = t(e)$. If $\{e, \bar{e}\}$ are nontrivial edges in (G, Y), then the fundamental group $\pi_1(H, X, v_1)$ is $G_{v_1} \star_{G_e} G_{v_2}$, which is nonabelian by corollary 3.1.3. But $\pi_1(H, X, v_1)$ is a subgroup of $\pi_1(G, Y, v_1)$ and every subgroup of an abelian group is abelian. At least one edge e' of $\{e, \bar{e}\}$ is trivial in (G, Y). Let (G', Y') be the graph of groups obtained by collapsing the trivial edge e' of the graph of groups (G, Y). In (G', Y'), Y' contains $n-1$ vertices.

□

Corollary 3.1.6. *Let (G, Y) be a graph of groups with a finite graph Y.*

1. *If $\pi_1(G, Y, v_0)$ is finite then $\pi_1(G, Y, v_0)$ is isomorphic to a vertex group G_v of (G, Y).*
2. *If $\pi_1(G, Y, v_0)$ is abelian then $\pi_1(G, Y, v_0)$ is either isomorphic to G_v or is isomorphic to $G_v *_{G_e}$ where G_v is a vertex group of (G, Y) and G_e is isomorphic to G_v.*

Proof. (1.) If $\pi_1(G, Y, v_0)$ is finite then $\pi_1(G, Y, v_0) \cong \pi_1(G', Y', v_0')$ such that (G', Y') is a reduced graph of groups where the graph Y' is a vertex v_0' with no edges. The group $\pi_1(G', Y', v_0')$ is isomorphic to $G_{v_0'}$ and $G_{v_0'}$ is isomorphic to a vertex group G_v of (G, Y).

(2.) If $\pi_1(G, Y, v_0)$ is abelian then $\pi_1(G, Y, v_0) \cong \pi_1(G', Y', v_0')$ such that (G', Y') is a reduced graph of groups where the graph Y' is a vertex v_0' with no edges or the graph Y' is a vertex v_0' with an edge e. The group $\pi_1(G', Y', v_0')$ is isomorphic to $G_{v_0'}$ or $G_{v_0'} *_{G_e}$. The vertex group $G_{v_0'}$ is isomorphic to a vertex group G_v of (G, Y).

Suppose that $G_{v_0'}$ is nontrivial. By Corollary 3.1.3, If the graph Y' is a vertex v_0' with an edge e and $f_e(G_e), f_{\bar{e}}(G_e)$ are proper subgroups then the group $G_{v_0'} *_{G_e}$ is nonabelian. Therefore the edge group G_e is isomorphic to $G_{v_0'}$. If $G_{v_0'}$ is trivial then $\pi_1(G', Y', v_0')$ is isomorphic to $\{\emptyset\}$ or $\{\emptyset\}*_{\{\emptyset\}}$.

□

3.2 Graph of groups of $\pi_1(X_\Gamma)$

A natural way of associating a graph of groups (G, Y) to $\pi_1(X_\Gamma)$ was given in [9], such that $\pi_1(G, Y, v_0)$ is isomorphic to $\pi_1(X_\Gamma)$. We review this construction since we are going to need it to determine the finite 2-stratifold groups and abelian 2-stratifold groups.

For a black vertex b representing a singular oriented circle C_b, let $o(b)$ be the order of C_b in $\pi_1(X_\Gamma)$. Note that, if e is an edge joining a black vertex b to a white vertex w and the label of e is m, then e represents an oriented circle c of ∂W whose order in $\pi_1(X_\Gamma)$ is $k = o(b)/(o(b), m)$. Here $(o(b), m)$ denotes the greatest common divisor of $o(b)$ and m. (If $o(b) = 0$, then $(o(b), m) = m$).

Construct a space $\hat{X}$ from X by attaching disks as follows: If b has finite order $o(b)$ then attach a 2-cell d_b to C_b such that d_b is attached by a map of degree $o(b)$. If e is an edge joining b to w with label m and $o(b) \geq 1$, attach to c a 2-cell d_e with degree $k = o(b)/(o(b), m)$. (If o(b) = 0, do not attach 2-cells).

The group $\pi_1(\hat{X})$ isomorphic to $\pi_1(X_\Gamma)$. The graph of spaces associated to $\hat{X}$ has the same underlying graph as Γ_X with vertices $\hat{X}_b$, $\hat{X}_w$, and edges $\hat{X}_e$, defined as follows:

$\hat{X}_b$: For a black vertex b of Γ_X, $\hat{X}_b = N(C_b) \cup d_b \cup (d_e)$, where e runs over the edges having b as an endpoint.

$\hat{X}_w$: For a white vertex w of Γ_X, $\hat{X}_w = W \cup (d_e)$, where e runs over the edges incident to w.

$\hat{X}_e$: For an edge e joining b to w, $\hat{X}_e = (\hat{X}_b \cap \hat{X}_w)$.

The spaces X_b, X_w and X_e are path-connected and the inclusion-induced homomorphisms $\pi_1(X_e) \to \pi_1(X_b)$ and $\pi_1(X_e) \to \pi_1(X_w)$ are injective. This graph of spaces determines a graph of groups (G, Y) where $\mathbf{Y} = \Gamma_X$ such that $\mathbf{Y}$ is the realization of Y. The graph Y is a bipartite graph which is induced by Γ_X. The vertex groups are $G_b = \pi_1(\hat{X}_b)$ and $G_w = \pi_1(\hat{X}_w)$, the edge groups are $G_e = \pi_1(\hat{X}_e)$, the monomorphisms $G_e \to G_b$ (resp. $G_e \to G_w$) are induced by inclusion. Then $\pi_1(G, Y, v_0) \cong \pi_1(\hat{X})$.

The groups G_b of the black vertices and the groups G_e of the edges are cyclic. The groups G_w of the white vertices with edges $e_1, \dots, e_p$ labelled $m_1, \dots, m_p$ have the following presentation,

$$G_w = \{c_1, \dots, c_p, y_1, \dots, y_n : c_1 \dots c_p q = 1,\, c_1^{m_1}, \dots, c_r^{m_r}\ (r \le p)\},$$

where $p, n \ge 0$ and $q = [y_1, y_2] \dots [y_{2g-1}, y_{2g}]$ or $q = y_1^2 \dots y_g^2$. If a group G has a presentation given by G_w where all $m_i \ge 2$ and $r = p$ then G is an **F-group**. If G has a presentation given by G_w and $n = 0$ such that $p = 2$, $1 \le r < p$ and $m_i \ge 2$ then G can be written as a finite cyclic group. Otherwise G_w is a free product of cyclic groups or is isomorphic to the fundamental group of $(p - r)$-punctured surface of genus $\pm g$.

3.3 F-groups

We will now review the finite F-groups. Then we will show that an abelian F-group is either finite cyclic, the dihedral group of order 4, or $\mathbb{Z} \times \mathbb{Z}$.

Consider $\mathcal{F}$ to be an F-group as above. The finite F-groups are determined in [13].

Lemma 3.3.1. *The group $\mathcal{F}$ is finite cyclic if and only if $n = 0$ and $p \le 2$ or $n = 1$ and $p \le 1$. The group $\mathcal{F}$ is finite non-cyclic if and only if $n = 0$, $p = 3$, and (m_1, m_2, m_3) is either $(2, 2, m)$ with $m \ge 2$ (dihedral group of order $2m$) or $(2, 3, k)$ with $3 \le k \le 5$ (the tetrahedral, octahedral, dodecahedral groups).*

We now determine the abelian F-groups.

It follows from [14] (pgs. 68, 71, 86.) that if $n = 0$, $p = 3$, and $m_i \ge 2$ then $\mathcal{F}$ is an index 2 subgroup of the triangle group $T(m_1, m_2, m_3)$. If $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} > 1$ then $\mathcal{F}$ is finite non-cyclic group as in the previous lemma. If $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1$ then $\mathcal{F}$ is finite index subgroup of a hyperbolic triangle group. Hyperbolic triangle groups are fuchsian groups. Hence $\mathcal{F}$ is a noncyclic fuchsian group. By [12], all abelian fuchsian groups are cyclic. If $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 1$ then (m_1, m_2, m_3) is $(2, 3, 6)$, $(2, 4, 4)$, or $(3, 3, 3)$ and the group $\mathcal{F}$ is infinite. The presentation of $\mathcal{F}$ which is

$$\{c_1, c_2, c_3 : c_1 c_2 c_3 = 1,\, c_1^{m_1}, c_2^{m_2}, c_3^{m_3}\},$$

can be rewritten as

$$\{c_1, c_2 : c_1^{m_1}, c_2^{m_2}, (c_1c_2)^{m_3}\}.$$

If N is the commuter subgroup of $\mathcal{F}$ then the group $\mathcal{F}/N$ has the presentation

$$\{c_1, c_2 : c_1^{m_1}, c_2^{m_2}, (c_1c_2)^{m_3}, [c_1, c_2]\}$$

,

which can be rewritten as

$$\{c_1, c_2 : c_1^{m_1}, c_2^{m_2}, c_1^{m_3}c_2^{m_3}, [c_1, c_2]\}.$$

Therefore $\mathcal{F}/N$ is either $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$. We conclude that if $n = 0$, $p = 3$, and $m_i \geq 2$ and $\mathcal{F}$ is abelian then $\mathcal{F}$ is the dihedral group of order 4.

If $n = 0$, $p > 3$, and $m_i \geq 2$ then $\mathcal{F}$ surjects onto an F-group where $n = 0$, $p = 3$. Then we assume that $m_i = 2$ for all i otherwise $\mathcal{F}$ surjects onto a nonabelian F-group. Further assume $p = 4$, since if $p > 4$ and $m_i = 2$ for all i then $\mathcal{F}$ surjects onto an F-group where $p = 4$ and $m_i = 2$. The group $\mathcal{F}$ is infinite. If N is the commuter subgroup of $\mathcal{F}$ then $\mathcal{F}/N$ is $\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$. In this case, if $n = 0$, $p > 3$ then $\mathcal{F}$ is not abelian.

If $n \geq 1$ and $p > 1$ then $\mathcal{F}$ is a free product of $\{c_1, \ldots, c_p | c_1^{m_1}, \ldots, c_p^{m_p}\}$ and $\{y_1, \ldots, y_n\}$ amalgamated along the infinite cyclic subgroup $< c_1 \ldots c_p = q^{-1} >$. For $n > 2$, the group $\mathcal{F}$ contains a nonabelian surface group. If $n = 2$ and $p = 1$ then $\mathcal{F}$ has the presentation $\{y_1, y_2 | (q)^{m_1}\}$. If $q = [y_1, y_2]$ then $y_1y_2y_1^{-1}y_2^{-1}$ is nontrivial. If $q = y_1^2y_2^2$ then $\mathcal{F}$ surjects onto the fundamental group of a klein bottle. If $n = 2$ and $p = 0$ then $\mathcal{G}$ is either the fundamental group of the 2-torus or the klein bottle. We conclude that if $n > 1$ and $\mathcal{F}$ is abelian that $n = 2$, $p = 0$ and $\mathcal{F}$ is $\mathbb{Z} \times \mathbb{Z}$.

Lemma 3.3.2. *The group $\mathcal{F}$ is a finite abelian group if and only $n = 0$ and $p \leq 2$, $n = 1$ and $p \leq 1$, or $n = 0$, $p = 3$, and (m_1, m_2, m_3) is $(2, 2, 2)$. The group $\mathcal{G}$ is a infinite abelian group if and only if $n = 2$, $p = 0$, and $q = [y_1, y_2]$.*

3.4 Finite 2-stratifold and Abelian 2-stratifold groups

In this section we will determine the finite 2-stratifold groups and abelian 2-stratifold groups. Before that we will study whether certain HNN groups are abelian.

First we determine the abelian groups that admit a presentation given by G_w. Consider $\mathcal{G}$ to be group with presentation G_w. If $1 \leq r < p$ and $m_i < 2$ then $\mathcal{G}$ is isomorphic to the fundamental group of $(p-r)$-punctured surface of genus $\pm g$. For $n > 2$, the group $\mathcal{G}$ surjects onto a nonabelian surface group. If $n = 2$ where $p \geq 2$, $1 \leq r < p$ and $m_i \geq 2$ then $\mathcal{G}$ is a free product of cyclic groups. Suppose that $n < 2$. If $n = 1$ such that $p \geq 2$, $1 \leq r < p$ and $m_i \geq 2$ then $\mathcal{G}$ is a nontrivial free product of cyclic groups. If $n = 0$ such that $p > 2$, $1 \leq r < p$ and $m_i \geq 2$ then $\mathcal{G}$ is a nontrivial free product of cyclic groups. If $n = 0$ such that $p = 2$, $1 \leq r < p$ and $m_i \geq 2$ then $\mathcal{G}$ is finite cyclic. Therefore if $\mathcal{G}$ is abelian and not an F-group then $\mathcal{G}$ is infinite cyclic. We note that $\mathcal{G}$ is possibly the trivial group $\{\emptyset\}$.

From the previous paragraph along with Lemma 3.3.1 and Lemma 3.3.2 we note the following.

Remark 3.4.1. *If $\mathcal{G}$ is finite (nontrivial) then $\mathcal{G}$ is finite cyclic, dihedral group of order $2n$ or either the tetrahedral, octahedral, dodecahedral group. If $\mathcal{G}$ is abelian (nontrivial) then $\mathcal{G}$ is either cyclic, dihedral group of order 4, or $\mathbb{Z} \times \mathbb{Z}$.*

Lemma 3.4.2. *Let H be a cyclic subgroup of G.*

1. *If G is infinite cyclic and $G\star_H$ is abelian then $G\star_H$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*
2. *If G is finite cyclic and $G\star_H$ is abelian then $G\star_H$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_n$ for $n \geq 2$.*
3. *If G is $\mathbb{Z} \times \mathbb{Z}$ then $G\star_H$ is nonabelian.*
4. *If G is $\mathbb{Z}_2 \times \mathbb{Z}_2$ then $G\star_H$ is nonabelian.*

Proof. If H is trivial then the HNN group $G\star_H$ is a free product of an infinite cyclic group and G. We assume that H is a nontrivial cyclic subgroup.

(1.) The HNN group $G\star_H$ is the Baumslag-Solitar group $BS(n,m)$. The group $BS(n,m)$ is abelian if and only if $n = m = 1$.

(2.) By Corollary 3.1.3, if H is a proper subgroup of G then the HNN group $G\star_H$ is nonabelian. Suppose that $H = G$ and $G =< a|a^k >$. The HNN group $G\star_H$ has the presentation

$$< a, t | a^k, ta^m t^{-1} = a^n >$$

where $gcd(k,m) = 1$ and $gcd(k,n) = 1$. This presentation is equivalent to

$$< a, t | a^k, tat^{-1} = a^r >$$

where $gcd(k, r) = 1$ (and r may possibly 1). We assume that r is reduced mod k (i.e. $1 \leq r < k$). If $k = 2$ then $G\star_H$ is abelian. Suppose that $k > 2$ and $r \neq 1$. Then the word $tat^{-1}a^{-1} = a^{r-1}$. The word $a^{r-1} \neq 1$ since $(r-1) \mod k$ is not congruent to k. The word $tat^{-1}a^{-1}$ is nontrivial. If $k > 2$ and $r \neq 1 \mod k$ then $G\star_H$ is not abelian. Therefore if $G_{\star H}$ is abelian and $k > 2$ then $r = 1 \mod k$ and $G\star_H$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_k$.

(3.) All cyclic subgroups of $\mathbb{Z} \times \mathbb{Z}$ are proper. By Corollary 3.1.3, the HNN group $G\star_H$ is nonabelian.

(4.) All cyclic subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are proper. By Corollary 3.1.3, the HNN group $G\star_H$ is nonabelian.

□

The HNN group $\{\emptyset\}\star_{\{\emptyset\}}$ of $\{\emptyset\}$ is $\mathbb{Z}$. We now prove the main theorem of the section.

Theorem 3.4.3. *Let X be a 2-stratifold.*

1. *If X has finite fundamental group then $\pi_1(X)$ is either trivial, finite cyclic, dihedral group of order $2n$, or the tetrahedral, octahedral, dodecahedral groups.*
2. *If X has abelian fundamental group then $\pi_1(X)$ is either trivial, cyclic, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z} \times \mathbb{Z}$, or $\mathbb{Z} \times \mathbb{Z}_n$.*

Proof. Suppose that (G, Y) is the associated graph of groups to $\pi_1(X_\Gamma)$ such that $\pi_1(G, Y, v_0) \cong \pi_1(X_\Gamma)$.

(1.) If (G, Y) is a graph of groups where $\pi_1(G, Y, v_0)$ is finite then $\mathbf{Y}$ is a tree, all vertex groups G_v and all edge groups G_e are finite. The vertex groups G_w of (G, Y) are finite F-groups. The vertex groups G_b and edge groups G_e of (G, Y) are finite cyclic groups.

By corollary 3.1.6, $\pi_1(G, Y, v_0)$ is isomorphic to a vertex group of (G, Y). Therefore $\pi_1(G, Y, v_0)$ is isomorphic to either the trivial group or a finite F-group.

(2.) If (G, Y) is a graph of groups where $\pi_1(G, Y, v_0)$ is abelian then $\mathbf{Y}$ is a tree or homotopy equivalent to S^1, all vertex groups G_v and all edge groups G_e are abelian. By remark 3.4.1, the vertex groups G_w of (G, Y) are either cyclic, $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z} \times \mathbb{Z}$. The vertex groups G_b and edge groups G_e of (G, Y) are cyclic groups.

Suppose that $\mathbf{Y}$ is a tree. By corollary 3.1.6, $\pi_1(G, Y, v_0)$ is isomorphic to a vertex group of (G, Y). Therefore $\pi_1(G, Y, v_0)$ is either the trivial group or is isomorphic to a cyclic group, $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z} \times \mathbb{Z}$.

Suppose that $\mathbf{Y}$ is homotopy equivalent to S^1. By corollary 3.1.6, $\pi_1(G, Y, v_0)$ is isomorphic to $G_v *_{G_v}$ where G_v is a vertex group of (G, Y). The only vertex groups of (G, Y) that are isomorphic to edge groups are cyclic groups. Therefore $\pi_1(G, Y, v_0)$ is isomorphic to $\{\emptyset\} *_{\{\emptyset\}}$, $\mathbb{Z}_n *_{\mathbb{Z}_n}$, or $\mathbb{Z} \star_{\mathbb{Z}}$. The HNN group $\{\emptyset\} *_{\{\emptyset\}}$ is $\mathbb{Z}$. By Lemma 3.4.2, If $\mathbb{Z}_n *_{\mathbb{Z}_n}$ and $\mathbb{Z} \star_{\mathbb{Z}}$ are abelian then $\mathbb{Z}_n *_{\mathbb{Z}_n} \cong \mathbb{Z} \times \mathbb{Z}_n$ and $\mathbb{Z} \star_{\mathbb{Z}} \cong \mathbb{Z} \times \mathbb{Z}$. Therefore $\pi_1(G, Y, v_0)$ is isomorphic to either $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}_n$, or $\mathbb{Z} \times \mathbb{Z}$.

□

Let X be a 2-stratifold. If $\pi_1(X)$ is finite then necessary conditions on Γ_X are given by Lemma 2.3.3. In [10], the necessary conditions on Γ_X so that $\pi_1(X) \cong \mathbb{Z}$ is given by Proposition 3 (pg. 6). We find further necessary conditions for Γ_X if $\pi_1(X)$ is isomorphic to either $\mathbb{Z} \times \mathbb{Z}_n$ or $\mathbb{Z} \times \mathbb{Z}$.

Before we find the further necessary conditions for Γ_X, we recall the following Lemma which was shown in [10].

Lemma 3.4.4. *Let X be a 2-stratifold where Γ_X is a tree.*

1. *If Γ_X has at most one black terminal vertex and all white vertices are of genus 0 then $H_1(X_\Gamma)$ is finite.*

2. *If Γ_X has no black terminal vertices, contains at most one white vertex of genus -1 and all other white vertices are of genus 0 then $H_1(X_\Gamma)$ is finite.*

Lemma 3.4.5. *Let X be 2-stratifold.*

1. *If $\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}_n$ then Γ_X is homotopy equivalent to S^1, all white vertices are genus 0, and all terminal vertices are white.*

2. *If $\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}$ then Γ_X is homotopy equivalent to S^1, all white vertices are genus 0, and all terminal vertices are white or Γ_X is a tree, all terminal vertices are white, and contains one white vertex of genus 1 while all other white vertices are genus 0.*

Proof. Suppose that Γ_X is a tree. By Lemma 3.4.4, if Γ_X has all white vertices of genus 0, and contains at most one black terminal vertex or Γ_X has no black terminal vertices, contains at most one white vertex of genus -1 and all other white vertices are of genus 0 then $H_1(X_\Gamma)$ is finite. Then by lemma 2.3.5 if $\pi_1(X)$ is abelian and infinite then Γ_X has all white terminal vertices and contains one white vertex of genus 1 while all other white vertices are genus 0. If $\pi_1(X)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_n$ and Γ_X contains one white vertex of genus 1 then $\mathbb{Z} \times \mathbb{Z}_n$ surjects onto $\mathbb{Z} \times \mathbb{Z}$. If $\pi_1(X)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ then Γ_X has all white terminal vertices and contains one white vertex of genus 1 while all other white vertices are genus 0.

Suppose that Γ_X is not a tree. By Lemma 2.3.5, if $\pi_1(X)$ is abelian and infinite then Γ_X is homotopy equivalent to S^1, all white vertices are genus 0, and all terminal vertices are white. It follows that if $\pi_1(X)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}_n$ then Γ_X is homotopy equivalent to S^1, all white vertices are genus 0, and all terminal vertices are white.

□

CHAPTER 4

GRAPHS OF TRIVALENT 2-STRATIFOLDS WITH FINITE FUNDAMENTAL GROUP

The goal of this chapter is to find further necessary conditions on the graph Γ_X of a trivalent 2-stratifold X so that $\pi_1(X)$ is finite.

We begin with the definition of a trivalent 2-stratifold X and a surgery on the graph Γ_X. This surgery, called operation $B1$, will be used many times in all the remaining chapters. We then find conditions on Γ_X that guarantee that X will have infinite fundamental group. From these conditions, we then determine the necessary conditions on Γ_X so that $\pi_1(X_\Gamma)$ is finite. These are given by Theorem 4.3.7.

4.1 Pruned trivalent 2-stratifolds

We review the definition of a trivalent 2-stratifold and the definition of a pruned trivalent 2-stratifold graph.

A 2-stratifold X is called **trivalent** if the graph Γ_X has every black vertex b either incident to three edges, each with label 1, two edges, one with label 1, the other with label 2, or b is a terminal vertex with adjacent edge of label 3. A graph Γ_X is also said to be **trivalent** if X_Γ is a trivalent 2-stratifold. A trivalent 2-stratifold that consists of one black vertex with all white vertices of genus 0 is called either a $b111$-tree, $b12$-tree, or a $b3$-tree if the black vertex has degree 3, 2, or 1 respectively. Closed 2-manifolds are considered to be trivalent 2-stratifolds.

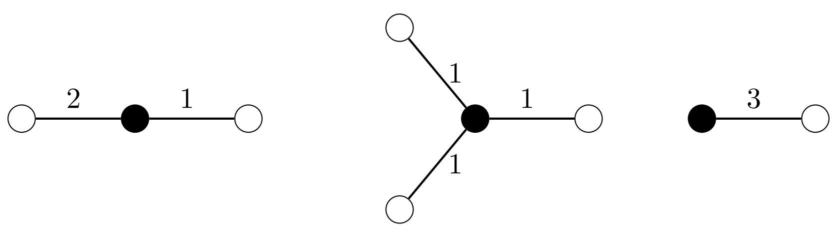


Figure 4.1: A $b12$-graph, a $b111$-graph, and a $b3$-graph.

We recall the definition of p-strings and q-strings, which were introduced in [10]. A **p-string** of length $2r$ is an oriented linear graph $w_0 - b_1 - w_1 - b_2 - ... - b_r - w_r$ with all white vertices w_i of genus 0, successive edge labels 1212...12 (starting at w_0) and with r labels of 2. A **q-string** is an oriented linear graph with all white vertices w_i of genus 0, successive edge labels 2121...21 (starting at w_0), and with r labels of 2. A p-string (or q-string) is **terminal** if w_r is a terminal white vertex of Γ.

If L is a terminal q-string then pruning L from Γ_X does not alter the fundamental group of a X. We say a trivalent 2-stratifold graph Γ is **pruned** if Γ contains no terminal q-strings. A trivalent 2-stratifold X is also said to be **pruned** if the associated labeled graph Γ_X is pruned.

4.2 Properties of trivalent 2-stratifold graphs

A useful operation on the graphs Γ_X of trivalent 2-stratifolds is introduced. This operation is a surgery on Γ_X that will produce a graph Γ' such that the fundamental groups $\pi_1(X_\Gamma)$ and $\pi_1(X_{\Gamma'})$ are isomorphic.

For trivalent 2-stratifolds X whose graph Γ_X contains $n > 1$ black vertices of degree 3, the operation $B1$, (seen below), applied to the graph Γ_X produces a new graph Γ' that contains $n-1$ black vertices of degree 3.

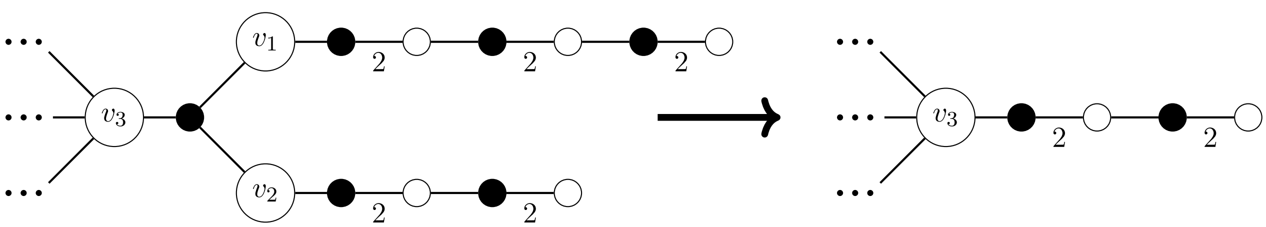


Figure 4.2: Operation B1

Let Γ be a trivalent graph containing a black vertex b of degree 3 with adjacent vertices v_1, v_2, v_3, such that v_i is the initial vertex of a terminal p-string P_i of length $2p_i$ for $i = 1, 2$. **Operation B1** produces a trivalent graph Γ' from Γ by replacing $st(b) \cup P_1 \cup P_2$ with a p-string P' (with initial vertex v_3) of length $\min\{2p_1, 2p_2\}$. The p-string P' in Γ' will be referred to as the **associated p-string**.

Lemma 4.2.1. *Let X be a trivalent 2-stratifold whose graph Γ_X contains $n > 1$ black vertices of degree 3. Let b to be a black vertex of Γ_X with degree 3 that is adjacent to the initial vertex of two terminal p-strings P_1, P_2 with length $2p_1, 2p_2$ respectively. Let Γ' be obtained from Γ by operation $B1$.*

Then $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$ and Γ' contains $n-1$ black vertices of degree 3.

Proof. L-prune the terminal p-strings P_i. In the resulting graph Γ', the black vertex b is adjacent to two terminal vertices v_1', v_2' where the edge incident to b and v_i' has label 2^{p_i}. L-pruning induces an isomorphism, so $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$. Let the terminal linear graph, whose initial vertex is b and whose terminal vertex is v_i', be called L_i

Construct Γ'' by replacing $(L_1 \setminus b) \cup (L_2 \setminus b)$ with a single terminal linear branch L'' of length 1, with initial vertex b, terminal vertex w of genus 0, and with edge label $min(2^{p_1}, 2^{p_2})$. The group $\pi_1(X_{\Gamma'})$ is isomorphic to $\pi_1(X_{\Gamma''})$ by Lemma 2.2.4.

The stratifold $X_{\Gamma''}$ is not a trivalent 2-stratifold. Replace the terminal linear graph $L'' \cup st(b) \cup v_3$ with a p-string P' of length $min(2p_1, 2p_2)$ with initial vertex which has been replaced by v_3. The resulting graph Γ''' contains $n-1$ black vertices of degree 3, $X_{\Gamma'''}$ is a trivalent 2-stratifold, and the fundamental group $\pi_1(X_{\Gamma'''})$ is isomorphic to $\pi_1(X_\Gamma)$. □

By inductively applying the operation $B1$, it will be shown that a trivalent 2-stratifold graph Γ_X will be produced with no black vertices of degree 3 if X has finite fundamental group. To insure this can be inductively done, we show that certain trivalent 2-stratifold graphs Γ_X have the property given in Corollary 4.2.3. Corollary 4.2.3 follows from the next lemma and we consider single white vertices as a linear graphs.

Lemma 4.2.2. *Suppose that Γ is a tree. If every nonterminal vertex of Γ has degree 3 then Γ contains two more terminal vertices than nonterminal vertices.*

Proof. Suppose the graph Γ has m total vertices then the number of edges is $m-1$ since Γ is a tree. If Γ contains k terminal vertices then the number of nonterminal vertices is $m-k$. By the handshaking lemma we have

$$k + 3(m-k) = 2(m-1).$$

The total number of vertices is then $m = 2k-2$. Therefore we get

$$(m-k) = k-2.$$

□

Corollary 4.2.3. *Let X be a trivalent 2-stratifold. If Γ_X is a tree that contains $n > 1$ black vertices of degree 3, all white vertices are degree ≤ 2 and no black terminal vertices then Γ_X contains at least two black vertices of degree 3 that are adjacent to the initial vertex of two terminal linear subgraphs.*

4.3 Graphs of trivalent 2-stratifolds

The necessary conditions for when a graph Γ_X represents certain pruned trivalent 2-stratifolds X with finite fundamental group is obtained. In this section, it is assumed that all 2-stratifolds X have an associated graph Γ_X that is a **tree** that satisfies one of the following conditions:

1. *The graph Γ_X has exactly one black terminal vertex, all other terminal vertices are white, and all white vertices are genus* 0.
2. *The graph Γ_X has exactly one white vertex of genus -1, all other white vertices are genus* 0, *and all terminal vertices are white.*
3. *The graph Γ_X has all white terminal vertices and white vertices are of genus* 0.

By Theorem 2.3.3, these are necessary conditions on X for X to have finite fundamental group.

Lemma 4.3.1. *Let X be a 2-stratifold. Denote a linear subgraph L of Γ_X with vertices $w_0 - b_1 - w_1$ and successive labels m, n as $L(m, n)$. The black vertex b_1 of L has degree 2 and the white vertices w_i of L are genus 0. Denote a linear subgraph L' of Γ_X with vertices $w_0 - b_1 - w_1 - b_2 - w_2$ and successive labels m_1, n_1, m_2, n_2 as $L'(m_1, n_1, m_2, n_2)$. The black vertex b_i of L' have degree 2 and the white vertices w_i of L' are genus 0.*

1. *If Γ_X contains a white vertex of genus -1 and a linear subgraph $L(m, n)$ where $k = gcd(m, n) > 1$ then $\pi_1(X)$ surjects onto $\mathbb{Z}_2 * \mathbb{Z}_k$.*
2. *If Γ_X contains at least two linear subgraphs $L_1(m_1, n_1)$, $L_2(m_2, n_2)$ where $k_i = gcd(m_i, n_i) > 1$ for $i = 1, 2$ then $\pi_1(X)$ surjects onto $\mathbb{Z}_{k_1} * \mathbb{Z}_{k_2}$.*
3. *If Γ_X contains a black terminal vertex whose incident edge has label $r > 2$ and a linear subgraph $L(m, n)$ where $k = gcd(m, n) > 1$ then $\pi_1(X)$ surjects onto $\mathbb{Z}_r * \mathbb{Z}_k$.*

4. If Γ_X contains two linear subgraphs $L_1(2,1,1,2)$, $L_2(2,1,1,2)$ then $\pi_1(X)$ surjects onto $\mathbb{Z}_2 * \mathbb{Z}_2$.

Proof. (1.) Allow w to be the white vertex of genus -1 and let b be the black vertex of the the linear subgraph $L(m,n)$ of Γ_X. Let T be the linear subgraph of Γ_X with terminal vertices w, b. Prune Γ_X at T. In the resulting graph, construct T' by attaching to each black vertex that is not b a white vertex of genus 0 with edge label 1. Then $\pi_1(X)$ surjects onto $\pi_1(X_{T'}) \cong \mathbb{Z}_2 * \mathbb{Z}_k$.

(2.) Allow b_i to be the black vertex for linear subgraph L_i of Γ_X for $i = 1, 2$. Further, let T be the linear subgraph of Γ_X with terminal vertices b_1, b_2 and prune Γ_X at T. In the resulting graph, construct T' by attaching to each black vertex not b_1 or b_2 a white vertex of genus 0 with edge label 1. Then $\pi_1(X)$ surjects onto $\pi_1(X_{T'}) \cong \mathbb{Z}_{k_1} * \mathbb{Z}_{k_2}$.

(3.) Allow b to be the black terminal vertex and let b' be the black vertex of the the linear subgraph L of Γ_X. Let T be the linear subgraph of Γ_X with terminal vertices b, b'. Prune Γ_X at T. In the resulting graph, construct T' by attaching to each black vertex not b or b' a white vertex of genus 0 with edge label 1. Then $\pi_1(X)$ surjects onto $\pi_1(X_{T'}) \cong \mathbb{Z}_r * \mathbb{Z}_k$.

(4.) This follows by a similar proof to (2.).

□

It should be noted that by the classification of simply connected trivalent 2-stratifolds, if X is a pruned trivalent 2-stratifold where Γ_X has all white vertices of genus 0, all terminal edges have label 2, and all terminal vertices are white then X is not simply connected. We highlight this fact below and use it to show certain trivalent 2-stratifold graphs Γ_X have an associated 2-stratifold X with infinite fundamental group.

Lemma 4.3.2. *Let X be a pruned trivalent 2-stratifold. If Γ_X has all white vertices of genus 0, all terminal edges have label 2, and all terminal vertices are white then X_Γ is not simply connected.*

Lemma 4.3.3. *Let X be a pruned trivalent 2-stratifold where the graph Γ_X has a label 2 for all edges incident to a terminal white vertex of genus 0. Then X has infinite fundamental group if Γ_X contains at least one of the following:*

1. *a black terminal vertex with edge label 3 and a white vertex of degree > 2;*
2. *a white vertex of genus -1 and a white vertex of degree > 2;*
3. *a white vertex of genus -1 with degree ≥ 2;*

4. or at least two white vertex w_1, w_2 of degree > 2.

Proof. (1.) Assume that b is the black terminal vertex of Γ_X and w is the white vertex of degree > 2. Let L be the linear subgraph of Γ_X with terminal vertices b, w. Suppose e is the edge in L incident to w. Let P be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus e$ that contains $L \setminus \{e, w\}$ and let K be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus e$ that contains w. If Γ_X is pruned at K, the resulting graph K' has a corresponding 2-stratifold $X_{K'}$ with nontrivial fundamental group $\pi_1(X_{K'})$ by Lemma 4.3.2. Now for the graph Γ_X, attach a white vertex of genus 0 with an edge of label 1 for all black vertices in P except b (see figure 4.3). Then there is an epimorphism from $\pi_1(X) \to \pi_1(X_{K'}) \star \mathbb{Z}_3$.

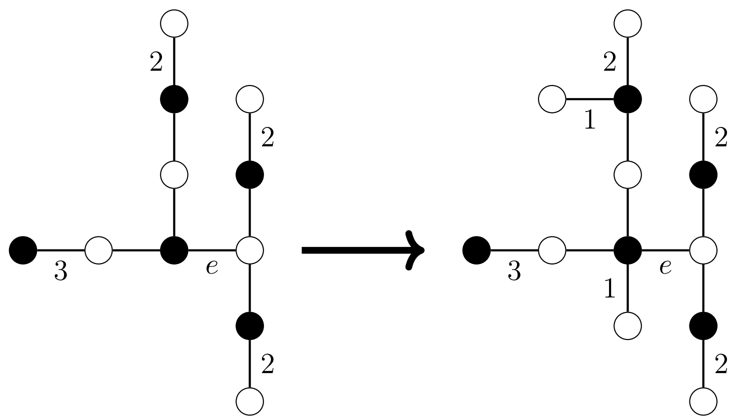


Figure 4.3: Lemma 4.3.3

(2.) Let v be a white vertex of genus -1 and w be a white vertex of degree 3. Let L be the linear subgraph of Γ_X with terminal vertices v, w. Suppose e is the edge in L incident to w. Let P be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus e$ that contains w. Prune Γ_X at $L \cup P$. The statement follows by a similar proof to (1.) on the resulting graph Γ'.

(3.) Suppose that Γ_X contains a white vertex v of degree 2 with genus -1 and assume all other white vertices are degree ≤ 2. (If Γ_X did contain a white vertex of genus 0 with degree >2 then by the previous part X has infinite fundamental group.)

Suppose that Γ_X has no black vertices of degree 3. The vertex v is not terminal and Γ_X is a linear graph. Let L_1 be the linear subgraph of Γ_X with initial vertex v and terminal vertex w where w is a terminal vertex of Γ_X. Orient the subgraph L_1 so that vertices are ordered as

$w_0^1 - b_1^1 - w_1^1 - b_2^1 - ... - b_r^1 - w_r^1$ with corresponding edge labels $m_1^1 - n_1^1 - \ldots - m_r^1 - n_r^1$ where $w_0^1 = v$ and $w_r^1 = w$. Similarly, let L_2 be the linear subgraph of Γ_X with initial vertex v and terminal vertex w' where w' is the other terminal vertex of Γ_X. Orient the subgraph L_2 so that vertices are ordered as $w_0^2 - b_1^2 - w_1^2 - b_2^2 - ... - b_l^2 - w_l^2$ with corresponding edge labels $m_1^2 - n_1^2 - \ldots - m_l^2 - n_l^2$ where $w_0^2 = v$ and $w_l^2 = w'$.

Suppose that at least one L_i contains a linear subgraph T with vertices $w_j^i - b_{j+1}^i - w_{j+1}^i - b_{j+2}^i - w_{j+2}^i$ and successive labels $2, 1, 1, 2$. If T is disjoint from v then $\pi_1(X)$ surjects onto $\mathbb{Z}_2 \star \mathbb{Z}_2$. If v is a terminal vertex of T then prune Γ_X at T. Note that, there is a surjection from $\pi_1(X_\Gamma)$ to $\pi_1(X_T)$. The group $\pi_1(X_T)$ admits the following presentation:

$$\{b_1, b_2, c, \gamma : b_1^2 = 1, b_1 = b_2, b_2^2 = c, c\gamma^2 = 1\}.$$

The group $\pi_1(X_T)$ is isomorphic $\mathbb{Z}_2 \star \mathbb{Z}_2$. Therefore if the subgraph L_i of Γ_X contains a linear subgraph T then $\pi_1(X)$ is infinite.

Suppose the labeling of L_i beginning with the edge incident to v is given by $12 \ldots 12$. Prune Γ_X at the linear subgraph $w_1^1 - b_1^1 - v - b_1^2 - w_1^2$. The resulting stratifold $X_{\Gamma'}$ has vertices $w_1^1 - b_1^1 - v - b_1^2 - w_1^2$ with successive edge labels, beginning at the edge incident to w_1^1, $2, 1, 1, 2$. The 2-stratifold $X_{\Gamma'}$ has a fundamental group that admits the following presentation:

$$\{b_1, b_2, \gamma : b_1^2 = 1, b_2^2 = 1, b_1 b_2 \gamma^2 = 1\}.$$

The group $\pi_1(X_{\Gamma'})$ surjects onto $\mathbb{Z}_2 \star \mathbb{Z}_2$.

Therefore for a graph Γ_X with no black vertices of degree 3 and a nonterminal white vertex of genus -1, the fundamental group of X_Γ is infinite.

Suppose that Γ contains one black vertex b of degree 3. The black vertex b is adjacent to the initial vertex w_1, w_2, w_3 of three terminal linear trees T_1, T_2, T_3 respectively. Let T_1 contain the white vertex v of genus -1 then T_2, T_3 contain only white vertices of genus 0. If either T_2, T_3 contains a subgraph $w_0 - b_1 - w_1 - b_2 - w_2$ with successive labels $2 - 1 - 1 - 2$ then $\pi_1(X_\Gamma)$ surjects onto $Z_2 * Z_2$. Otherwise, If T_2, T_3 are p-strings then apply operation $B1$ to $st(b) \cup T_2 \cup T_3$. The resulting graph Γ' is a linear 2-stratifold with a nonterminal white vertex of genus -1. $X_{\Gamma'}$ has infinite fundamental group and $\pi_1(X_\Gamma) \cong \pi_1(X'_\Gamma)$.

By induction, we assume that if Γ_X contains $k-1>0$ black vertices of degree 3 and a nonterminal white vertex of genus -1 then $\pi_1(X_\Gamma)$ is infinite.

Now assume Γ_X contains $k>0$ black vertices of degree 3 and a nonterminal white vertex v of genus -1. Let b be a black vertex of degree 3 that is adjacent to the vertices w_1, w_2, w_3 such that w_i is the initial vertex of a terminal linear subgraph T_i for $i=1,2$. (The black vertex b is an outermost such vertex, in that at least two components of $\Gamma_X \setminus st(b)$ contains only vertices with degree <3.) If v is contained in either T_1 or T_2, then by corollary 4.2.3, there exists another outermost black vertex b' of degree 3 that is adjacent to the initial vertex of two terminal linear branches that does not contain v. We assume that v is not contained in T_i. If there is a linear subgraph T with vertices $w_j - b_{j+1} - w_{j+1} - b_{j+2} - w_{j+2}$ and successive labels $2,1,1,2$ contained in some T_i then there is a surjection from $\pi_1(X)$ onto $\mathbb{Z}_2 \star \mathbb{Z}_2$. If T_i are p-strings then apply operation $B1$ on $st(b) \cup T_1 \cup T_2$ such that the resulting graph Γ' has $k-1$ black vertices of degree 3 and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$. The result follows.

(4.) Suppose that Γ_X has two white vertices w_1, w_2 of degree >2. Let L be a linear subgraph of Γ_X with terminal vertices w_1, w_2. Let e_1 and e_2 be the edges incident to w_1 and w_2 respectively contained in L. Let P be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus \{e_1, e_2\}$ that contains neither w_1 or w_2. Allow K_i be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus e_i$ that contains w_i. If Γ_X is pruned at K_i, the resulting graph K_i' has a corresponding 2-stratifold $X_{K_i'}$ with nontrivial fundamental group $\pi_1(X_{K_i'})$ by Lemma 4.3.2. Now for the graph Γ_X, attach a white vertex of genus 0 with edge label one to each black vertex in the subgraph P. Then $\pi_1(X)$ surjects onto $\pi_1(X_{K_1'}) \star \pi_1(X_{K_2'})$.

□

The next corollary follows from the proof of part (3.) of the previous lemma.

Corollary 4.3.4. *If X is a pruned trivalent 2-stratifold whose graph Γ_X has a white terminal vertex of genus -1 and all edges incident to a terminal vertex have label 2 then $\pi_1(X)$ has infinite fundamental group.*

We note that corollary 4.3.4 is not true if we alter the condition on the terminal edge labels.

Remark 4.3.5. *For a pruned trivalent 2-stratifold X whose graph Γ_X has a white terminal vertex of genus -1 and all edges incident to a terminal vertex of genus 0 have label 2, $\pi_1(X)$ need not be infinite.*

For example, a trivalent linear 2-stratifold $w_0 - b_1 - w_1 - b_2 - w_3$ with successive labels $1, 2, 1, 2$, where w_0 has genus -1 and w_1, w_2 have genus 0, has fundamental group $\mathbb{Z}_8$.

The figure below is an example of a horned tree. The main property of a horned tree H_T is that $\pi_1(X_{H_T})$ is isomorphic to $\mathbb{Z}_2$. We review the definition of a horned tree.

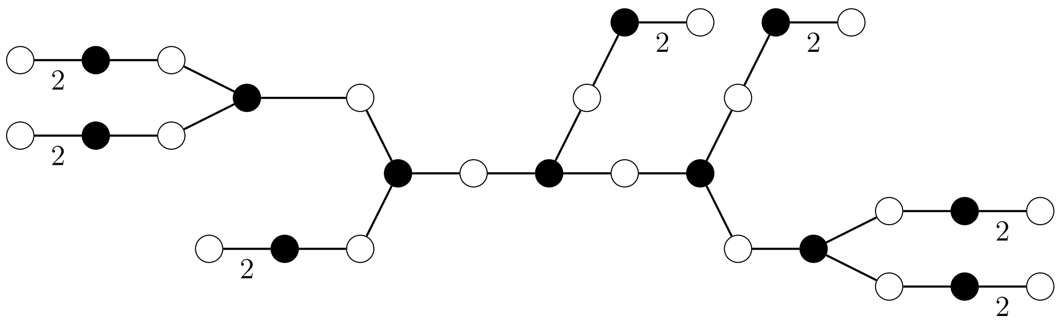


Figure 4.4: An example of a horned tree.

A **horned tree** H_T is a finite connected bipartite labelled tree such that

1. all white vertices are genus 0;
2. every black vertex b whose distance to a terminal white vertex is 1 has degree 2; otherwise b has degree 3;
3. every nonterminal white vertex has degree 2;
4. every terminal edge has label 2, every nonterminal edge has label 1;
5. there is at least one vertex of degree 3.

A trivalent linear 2-stratifold $w_0 - b_1 - w_1 - b_2 - w_3$ with sucessive labels $2, 1, 1, 2$ and all white vertices of genus 0 will be considered a horned tree.

To construct a horned tree (with black vertices of degree 3) take a connected finite tree composed of only $b111$-trees, delete the terminal vertices of this tree, and attach a $b12$-tree to each terminal edge so that 2 is the terminal edge label in the resulting graph. The graph obtained is a horned tree.

Lemma 4.3.6. *Let X be a pruned trivalent 2-stratifold where the graph Γ_X has a label 2 for all edges incident to a terminal white vertex of genus 0. Then X has infinite fundamental group if Γ_X contains one of the following:*

1. *a white vertex v of genus -1 and a horned tree H_T such that v and H_T are disjoint;*
2. *two horned trees H_1, H_2 where H_1 and H_2 are disjoint or H_1 and H_2 intersect at a vertex v such that $v = H_1 \cap H_2$;*
3. *a black terminal vertex with edge label 3 and a horned tree H_T;*
4. *a white vertex w of degree > 2 and a horned tree H_T such that either w and H_T are disjoint or w is a terminal vertex of H_T;*
5. *or a white vertex of degree > 3.*

Proof. (1.) Suppose that v and H_T are disjoint. By Lemma 4.3.3, v is a terminal vertex otherwise X has infinite fundamental group. Attach to each black vertex not contained in H_T a white vertex of genus 0 with edge label 1. Then there is an epimorphism from $\pi_1(X) \to \mathbb{Z}_2 \star \mathbb{Z}_2$.

(2.) Suppose that H_1 and H_2 are horned trees contained in the graph Γ_X. Attach to each black vertex not contained in H_1, H_2 of Γ_X a white vertex of genus 0 with edge label 1. Then there is an epimorphism from $\pi_1(X) \to \mathbb{Z}_2 \star \mathbb{Z}_2$.

(3.) Suppose that b is the black terminal vertex. Attach to each black vertex not contained in H_T or b a white vertex of genus 0 with edge label 1. There is an epimorphism from $\pi_1(X) \to \mathbb{Z}_2 \star \mathbb{Z}_3$.

(4.) Assume that w has degree equal to 3, all other white vertices are of degree < 3, and all white vertices have genus 0. The two main cases of this proof is when H_T is disjoint from w and when w is a terminal vertex of H_T.

Suppose that H_T is disjoint from w. Let L be the linear subgraph of Γ_X with terminal vertices w and v where v is a terminal vertex of H_T such that $L \cap H_T = v$. Let e_1, e_2 be the edges incident to w, v (respectively) that are contained in L. Allow the subgraph P to be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus \{e_1, e_2\}$ that contains $L \setminus \{e_1, e_2, w, v\}$. Also allow the subgraph R to be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus \{e_1\}$ that contains w. If Γ_X is pruned at R, the resulting graph R' has a corresponding 2-stratifold $X_{R'}$ with nontrivial fundamental group $\pi_1(X_{R'})$ by lemma 4.3.2. Prune Γ_X at $R \cup e_1 \cup e_2 \cup P \cup H_T$ and attach white vertices of genus 0 with edge label 1 to all black vertices contained in P of the pruned graph. The resulting graph Γ' has a fundamental group isomorphic to $\pi_1(X_{R'}) \star \pi_1(\mathbb{Z}_2)$.

Now suppose that w is a terminal vertex of H_T and let e_1, e_2 be the edges incident to w that are not contained in H_T. Allow the subgraph of Γ_X corresponding to the component of $\Gamma_X \setminus e_i$ that does not contain H_T be called D_i. Let $E_i = D_i \cup e_i \cup w$. By part (2.), if E_i contains a horned tree then $\pi_1(X)$ is infinite, so we assume that E_i contains no horned trees. Prune Γ_X at $E_1 \cup E_2 \cup H_T$ and let the resulting graph be called Γ'. We now show that the fundamental group of $X_{\Gamma'}$ is infinite. Therefore the fundamental group of X_Γ will be infinite.

If Γ' contains no black vertices of degree 3 then Γ' has a single white vertex w of degree 3 where w is a terminal vertex of H_T and w is the initial vertex of two terminal p-strings E_1, E_2 of length $2p, 2q$. The associated 2-stratifold $X_{\Gamma'}$ has fundamental group that can be represented with the following presentation:

$$\{c_1, c_2, c_3 : c_1^{2^p} = 1, c_2^{2^q} = 1, c_3^2 = 1, c_1c_2c_3^2 = 1\}.$$

The fundamental group $\pi_1(X_{\Gamma'})$ surjects onto $\mathbb{Z}_2 \star \mathbb{Z}_2$. Therefore if Γ' contains no black vertices of degree 3 then the fundamental group of $X_{\Gamma'}$ is infinite.

We proceed by induction. Assume that if Γ' contains $k-1 > 0$ black vertices of degree 3 then $\pi_1(X_{\Gamma'})$ is infinite.

Suppose that Γ' has $k > 0$ black vertices of degree 3. Let b be a black vertex of degree 3 that is adjacent to the vertices w_1, w_2, w_3 such that v_i is the initial vertex of a terminal linear subgraph T_i for $i = 1, 2$. (The black vertex b is an outermost such vertex, in that at least two components of $\Gamma_X \setminus st(b)$ contains only vertices with degree < 3.) If the terminal linear graphs T_i are contained in E_i or H_T then they are p-strings. Apply operation $B1$ on $st(b) \cup T_1 \cup T_2$ such that the resulting graph Γ'' has $k-1$ black vertices of degree 3 and $\pi_1(X'_\Gamma) \cong \pi_1(X_{\Gamma''})$. The result follows.

(5.) Suppose that w is the white vertex of degree 4 contained in Γ_X. Then Γ_X contains all white terminal vertices and all white vertices of genus 0, otherwise X has infinite fundamental group.

Suppose that Γ_X has no black vertices of degree 3. Let e_i be the edges incident to w for $1 \leq i \leq 4$. Define L_i to be the linear subgraph whose intial vertex is w, whose terminal vertex is a terminal vertex of Γ_X, and L_i contains the edge e_i. If at least one L_i contains a horned tree then X_Γ has infinite fundamental group. Assume then that each L_i is a p-string of length $2p_i$. The 2-stratifold X_Γ has fundamental group that can be represented with the following presentation:

$$\{c_1, c_2, c_3, c_4 : c_1^{2^{p_1}} = 1, c_2^{2^{p_2}} = 1, c_3^{2^{p_3}} = 1, c_4^{2^{p_4}} = 1, c_1c_2c_3c_4 = 1\}.$$

This is an infinite F-group.

Suppose that Γ_X has $k > 0$ black vertices of degree 3. Let b be a black vertex of degree 3 that is adjacent to the vertices w_1, w_2, w_3 such that v_i is the initial vertex of a terminal linear subgraph T_i for $i = 1, 2$. (The black vertex b is an outermost such vertex, in that at least two components of $\Gamma_X \setminus st(b)$ contains only vertices with degree < 3.) If T_i contains a horned tree then X_Γ has infinite fundamental group. We assume that the terminal linear subgraphs T_i are p-strings. Apply operation $B1$ on $st(b) \cup T_1 \cup T_2$ such that the resulting graph Γ' has $k-1$ black vertices of degree 3 and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$. The result follows by the induction hypothesis.

□

Theorem 4.3.7. *Let X be a pruned trivalent 2-stratifold where the graph Γ_X has a label 2 for all edges incident to a terminal white vertex of genus 0. If X has finite fundamental group then Γ_X is a tree that satisfies one of the following conditions:*

1. *Γ_X has one terminal vertex v of genus -1 whose incident edge label is 1 while all other white vertices are genus 0, all terminal vertices are white, all white vertices are of degree ≤ 2, and Γ_X contains no horned trees;*

2. *Γ_X has all white vertices of genus 0, all terminal vertices are white, and there is exactly one white vertex v of degree 3 while all other white vertices are of degree < 3, and Γ_X contains no horned tree H_T such that either v and H_T are disjoint or v is a terminal vertex of H_T;*

3. *Γ_X has all white vertices are genus 0, all terminal vertices are white, all white vertices are of degree ≤ 2, and Γ_X contains at most one horned tree;*

4. *Γ_X has all white vertices are genus 0, one black terminal vertex, all white vertices are of degree ≤ 2, and Γ_X contains no horned tree.*

Proof. By lemma 2.3.3, if X has finite fundamental group then the graph Γ_X is a tree that satisfies one of the following conditions:

1. The graph Γ_X has exactly one black terminal vertex, all other terminal vertices are white, and all white vertices are genus 0.
2. The graph Γ_X has exactly one white vertex of genus -1, all other white vertices are genus 0, and all terminal vertices are white.
3. The graph Γ_X has all white terminal vertices and white vertices are of genus 0.

If Γ_X contains exactly one white vertex v of genus -1 then v is terminal by Lemma 4.3.3 and the label incident to v is 1 by Corollary 4.3.4. Further, all white vertices of Γ_X are of degree < 3 by Lemma 4.3.3 and Γ_X contains no horned trees by Lemma 4.3.6.

If Γ_X contains all white vertices of genus 0 and all terminal vertices are white then there exists at most one white vertex v of degree > 2 by Lemma 4.3.3. If all white vertices of Γ_X are of degree < 3 then Γ_X contains at most one horned tree by Lemma 4.3.6. If Γ_X contains a white vertex v of degree > 2 then v is degree 3 and Γ_X contains no horned tree H_T such that either v and H_T are disjoint or v is a terminal vertex of H_T by Lemma 4.3.6.

If Γ_X contains exactly one black terminal vertex then Γ_X must have all white vertices of degree < 3 by Lemma 4.3.3 and Γ_X cannot contain a horned tree H_T by Lemma 4.3.6.

□

CHAPTER 5

LABELLINGS OF TRIVALENT 2-STRATIFOLDS WITH FINITE FUNDAMENTAL GROUP

A classification of all trivalent labelled graphs that represent simply connected trivalent 2-stratifolds was given in [7]. Then a classification of all trivalent labelled graphs that represent trivalent 2-stratifolds with infinite cyclic fundamental group was given in [10]. The approach in the infinite cyclic case was to find necessary and sufficient conditions on Γ_X such that $\pi_1(X) \cong \mathbb{Z}$. In the previous chapter we found the necessary conditions on Γ_X so that X_Γ has finite fundamental group. In this chapter we show that the necessary conditions on Γ_X that allow X_Γ to have finite fundamental group are sufficient conditions.

We first introduce a linear subgraph of Γ_X called an O-string (Order string). For a trivalent 2-stratifold X with finite fundamental group, O-strings will be used to determine the order of the fundamental group. Then for X where Γ_X satisfies the necessary conditions in Theorem 4.3.7, we compute the finite fundamental groups for X and the labellings of the associated graph Γ_X. This is done in lemma 5.1.3, lemma 5.1.5, lemma 5.1.6, and lemma 5.1.7.

Core-reduced subgraphs were introduced in [10] for graphs Γ_X that are homotopy equivalent to S^1. These graphs were important in the classification of trivalent 2-stratifolds with infinite cyclic fundamental group. We define core-reduced subgraphs for Γ_X where Γ_X is a tree. Then we use core-reduced subgraphs and lemmas 5.1.3, 5.1.5-5.1.7 to obtain a classification of trivalent labelled graphs that represent trivalent 2-stratifolds with finite fundamental group. This classification is given by corollaries 5.2.3-5.2.7.

5.1 Labellings of trivalent 2-stratifolds

In this section we compute the finite fundamental groups of the 2-stratifolds X_Γ whose associated bipartite labelled graphs Γ satisfy the necessary conditions given by Theorem 4.3.7.

The figure below is an example of a graph Γ that satisfies a set of conditions given by Theorem 4.3.7. The fundamental group of X_Γ is $\mathbb{Z}_{16}$. The order of this fundamental group is determined by the linear subgraph with initial vertex given by the genus -1 vertex and terminal vertex given by t_1. The connected subgraphs of Γ that are composed of red edges along with incident vertices are terminal p-strings. We use this example as motivation for the definition of an O-string.

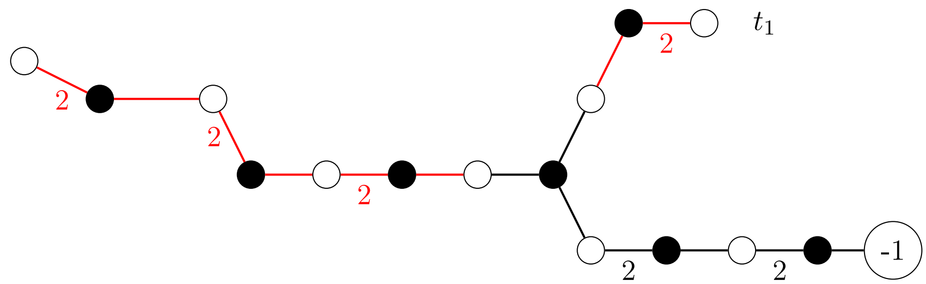


Figure 5.1: The graph Γ.

An O**-string** of length $2r$ is an oriented linear graph $w_0 - b_1 - w_1 - b_2 - ... - b_r - w_r$ where the genus of w_0 is either 0 or -1 while all other white vertices w_i are of genus 0, the labels m_i, n_i for the successive edges of $w_{i-1} - b_i - w_i$ are either $m_i = 1, n_i = 1$ or $m_i = 1, n_i = 2$ for $0 < i < r$, and the labels m_r, n_r for the edges of $w_{r-1} - b_r - w_r$ are given by the labels $m_r = 1, n_r = 2$. We note that terminal p-strings are O-strings.

Lemma 5.1.2 observes that certain subgraphs of a given O-string are preserved under operation $B1$. For example, the graph Γ' below is obtained by applying operation $B1$ to the graph Γ in the above figure. The linear subgraph with initial vertex given by the genus -1 vertex and terminal vertex given by t_1 is an O-string in both Γ and Γ' and contains the same number of edges with label 2. The subgraph composed of red edges and incident vertices in Γ' is the terminal associated p-string in Γ'.

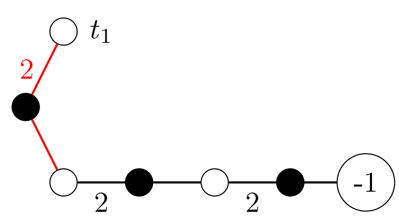


Figure 5.2: The graph Γ' obtained from applying operation $B1$ to Γ.

Before lemma 5.1.2, we introduce some notation and observe a fact about operation $B1$ and the graphs Γ_X, Γ'.

Remark 5.1.1. *Consider X to be a trivalent 2-stratifold where Γ_X contains a black vertex b of degree 3 with adjacent vertices v_1, v_2, v_3, such that v_i is the initial vertex of a terminal p-string P_i for $i = 1, 2$. Let Γ' be obtained from Γ by operation $B1$ at $st(b) \cup P_1 \cup P_2$. Let P' be the associated p-string in Γ'. The operation $B1$ does not alter $\Gamma_X \setminus (st(b) \cup P_1 \cup P_2)$. Then $\Gamma' \setminus (P' \setminus v_3) = \Gamma_X \setminus (st(b) \cup P_1 \cup P_2)$.*

For convenience, if v is a vertex of Γ' that is contained in $\Gamma' \setminus (P' \setminus v_3)$ then the same vertex in Γ_X contained in $\Gamma_X \setminus (st(b) \cup P_1 \cup P_2)$ will also be called v and vice versa. Similarly, if L is a linear subgraph of Γ' that is disjoint from $P' \setminus v_3$ with initial vertex v and terminal vertex w then the linear subgraph with initial vertex v and terminal vertex w contained in Γ_X that is disjoint from $st(b) \cup P_1 \cup P_2$ will also be called L and vice versa. Whether such an L is a subgraph of Γ' or a subgraph of Γ_X will be determined by context.

In general since $\Gamma' \setminus (P' \setminus v_3) = \Gamma_X \setminus (st(b) \cup P_1 \cup P_2)$, if S is a subgraph of Γ' that is contained in $\Gamma' \setminus (P' \setminus v_3)$ then the same subgraph in Γ_X contained in $\Gamma_X \setminus (st(b) \cup P_1 \cup P_2)$ will also be called S and vice versa.

We note that if L is an O-string in Γ' that is disjoint from $P' \setminus v_3$ then L is an O-string in Γ_X that is disjoint from $st(b) \cup P_1 \cup P_2$.

Lemma 5.1.2. *Let X be a trivalent 2-stratifold whose graph Γ_X is a tree that contains $n \geq 1$ black vertices of degree 3. Let b be a black vertex of degree 3 with adjacent vertices v_1, v_2, v_3, such that v_i is the initial vertex of a terminal p-string P_i of length $2p_i$ for $i = 1, 2$. Let Γ' be obtained from Γ by operation $B1$ at $st(b) \cup P_1 \cup P_2$. Let P' be the associated p-string in Γ'.*

Let L_i be a linear subgraph of Γ_X with an initial vertex w which is a white vertex not contained in P_i and a terminal vertex t_i where t_i is the terminal vertex of P_i and a terminal vertex of Γ_X. Let L' be a linear subgraph of Γ' with initial vertex w not contained in $P' \setminus w_3$ and terminal vertex t' where t' is the terminal vertex of P' and a terminal vertex of Γ'.

1. *If L' is an O-string then L_1, L_2 are O-strings.*
2. *If L' is an O-string that contains k edges with label 2 then L_1, L_2 contains $r \geq k$ edges with label 2 and at least one L_i has k edges with label 2.*
3. *If Γ' contains a horned tree $H_{T'}$ then Γ_X contains a horned tree H_T.*

4. *If a horned tree $H_{T'}$ of Γ' contains a terminal vertex of Γ' then a horned tree H_T of Γ_X contains a terminal vertex of Γ_X.*

Proof. (1.) Suppose L' is an O-string. Let S be the linear subgraph $w_0 - b_1 - w_1 - b_2 - ... - b_r - w_r$ of L' with initial vertex $w_0 = w$ and terminal vertex $w_r = v_3$. For $1 \leq i \leq r$, the labels m_i, n_i for the successive edges of $w_{i-1} - b_i - w_i$ contained in S are either $m_i = 1, n_i = 1$ or $m_i = 1, n_i = 2$. Let N_i be the linear subgraph of L_i with initial vertex v_3 and terminal vertex t_i. The subgraph N_i is an O-string. The subgraph L_i is composed of the subgraph S with initial vertex w and terminal vertex v_3 followed by the subgraph N_i with initial vertex v_3 and terminal vertex t_i. The linear graph L_i is an O-string.

(2.) Suppose that L' is an O-string that contains k edges with label 2. Let S and N_i be linear subgraphs as definied in (1.). By the previous proof L_i is an O-string. The subgraph S has $r' \geq 0$ edges with label 2. The subgraph P' has k' edges with label 2 where $k' + r' = k$. The integer k' is the minimum of $\{p_1, p_2\}$. Therefore for some i, N_i has k' edges with label 2. Then the linear graph L_i has $k' + r' = k$ edges with label 2.

(3.) Suppose Γ' contains a horned tree $H_{T'}$. For the terminal p-string P' of Γ', order the vertices $w'_0 - b'_1 - w'_1 - b'_2 - ... - b'_r - w'_r$ so that the initial vertex w'_0 is v_3 and w'_r is the terminal vertex t' of Γ'. The horned tree $H_{T'}$ is disjoint from P' or intersects P'. If the horned tree $H_{T'}$ is disjoint from P' then $H_{T'}$ is contained in Γ_X.

Suppose that $H_{T'}$ intersects P'. Then $H_{T'}$ intersects P' at only the vertex v_3 or along the linear subgraph P'' with initial vertex v_3 and terminal vertex w'_1. The linear subgraph P'' has vertices $w'_0 - b'_1 - w'_1$ where $w'_0 = v_3$ and successive labels $1, 2$. If the horned tree $H_{T'}$ intersects the subgraph of P' only at v_3 then $H_{T'}$ is contained in Γ_X. Suppose that the horned tree $H_{T'}$ contains the subgraph P'' of P'. Let H be a subgraph of $H_{T'}$ where $H = H_{T'} \setminus (st(b'_1) \cup w'_1)$. Then H is contained in Γ_X. For the terminal p-strings P_i of Γ_X, order the vertices $w^i_0 - b^i_1 - w^i_1 - b^i_2 - ... - b^i_{r_i} - w^i_{r_i}$ where $w^i_0 = v_i$ and $w^i_{r_i} = t_i$ of Γ_X for $i = 1, 2$ and define E_i to the linear subgraph of Γ_X with initial vertex v_3 and terminal vertex w^i_1. Then $H \cup E_1 \cup E_2$ is a horned tree contained in Γ_X.

(4.) Suppose Γ' contains a horned tree $H_{T'}$ where $H_{T'}$ contains a terminal vertex of Γ'. Let w be a terminal vertex of Γ' that is contained in $H_{T'}$. If P' is disjoint from $H_{T'}$ then $H_{T'}$ is contained in Γ_X and w is a terminal vertex of Γ_X and $H_{T'}$. We assume that P' is not disjoint from $H_{T'}$.

Suppose that w is disjoint from P'. Let H be the subgraph of $H_{T'}$ as defined in part (3.). The vertex w is contained in H and either $H_{T'}$ is contained in Γ_X or the horned tree $H_T = H \cup E_1 \cup E_2$ is contained in Γ_X where H, E_i are defined as in part (3.). If $H_{T'}$ is contained in Γ_X then w is a terminal vertex of Γ_X and $H_{T'}$. If H_T is contained in Γ_X then w is a terminal vertex of Γ_X and H_T.

Suppose that w is contained in P'. Then P' is a p-string of length 2 with initial vertex v_3 and terminal vertex w. It follows from (2.) that at least one of the terminal linear branches P_i in Γ_X is p-string of length 2. The horned tree $H \cup E_1 \cup E_2$ contains a terminal vertex of Γ_X.

□

Lemma 5.1.3. *Let X be a pruned trivalent 2-stratifold where Γ_X has a label 2 for all edges incident to a terminal white vertex of genus 0. Let Γ_X have all white vertices of genus 0, all terminal vertices are white, and all white vertices are of degree ≤ 2. If $\pi_1(X)$ is finite then all of the following hold:*

1. *Γ_X contains a horned tree H_T.*
2. *If L is a linear subgraph of Γ_X whose initial vertex v is a terminal vertex of H_T and whose terminal vertex w is a terminal vertex of Γ_X where $L \cap H_T = v$ and $w \neq v$ then L is an O-string.*
3. *The fundamental group $\pi_1(X)$ is isomorphic to $\mathbb{Z}_{2k+1}$ where the integer $k = 0$ if H_T contains a terminal vertex of Γ_X and $k > 0$ otherwise. The integer $k > 0$ corresponds to the minimal number of edges with label 2 in all linear subgraphs L whose initial vertex v is a terminal vertex of H_T and whose terminal vertex w is a terminal vertex of Γ_X where $L \cap H_T = v$ and $w \neq v$.*

Proof. By theorem 4.3.7, the fundamental group $\pi_1(X)$ is finite implies that the graph Γ_X is a tree that contains at most one horned tree.

Suppose that Γ_X has no black vertices of degree 3. The graph Γ_X is a linear graph. Orient the graph Γ_X so that vertices are ordered as $w_0 - b_1 - w_1 - b_2 - ... - b_r - w_r$ with corresponding edge labels $m_1 - n_1 - \ldots - m_r - n_r$. By assumption the subgraph $w_0 - b_1 - w_1$ has successive labels $m_1 = 2, n_1 = 1$ and the subgraph $w_{r-1} - b_r - w_r$ has successive labels $m_r = 1, n_r = 2$. Each subgraph $w_{i-1} - b_i - w_i$ for $1 < i < r$ has successive labels $m_i = 2, n_i = 1$ or $m_i = 1, n_i = 2$. There exists a j, where $1 < j \leq r$, such that $w_{j-2} - b_{j-1} - w_{j-1}$ has successive labels $m_{j-1} = 2, n_{j-1} = 1$ and $w_{j-1} - b_j - w_j$ has successive labels $m_j = 1, n_j = 2$. The graph Γ_X contains a horned tree H given by the graph $w_{j-2} - b_{j-1} - w_{j-1} - b_j - w_j$. By lemma 4.3.6, Γ_X does not contain any other horned tree.

Suppose H does not contain a vertex that is terminal in Γ_X. Let L_1 be the linear subgraph of Γ_X with initial vertex w_{j-2} and terminal vertex w_0 and let L_2 be the linear subgraph of Γ_X with initial vertex w_j and terminal vertex w_r. The linear subgraphs L_1, L_2 are p-strings of length $2p_1, 2p_2$. Otherwise Γ_X contains more than one horned tree. Note that L_1, L_2 are O-strings. L-prune Γ_X at the linear subgraphs L_1 and L_2. The resulting graph Γ' is a linear graph where $\Gamma' = \Gamma'(2^{p_1}, 1, 2, 1, 1, 2, 1, 2^{p_2})$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$. A presentation of the fundamental group of $X_{\Gamma'}$ is given by:

$$\{x_1, x_2, x_3, x_4 : x_1^{2^{p_1}} = 1, x_1 = x_2^2, x_2 = x_3, {x_3}^2 = x_4, x_4^{2^{p_2}} = 1\}.$$

This presentation is equivalent to:

$$\{x_3 : x_3^{2^{p_1+1}} = 1, x_3^{2^{p_2+1}} = 1\}.$$

This group is finite cyclic of order given by the $\min(2^{p_1+1}, 2^{p_2+1})$. Therefore $\pi_1(X) \cong \mathbb{Z}_{2^{k+1}}$ where k is the minimum of $\{p_1, p_2\}$. The number k is the minimum number of edges with label 2 in the O-strings L_1, L_2.

Suppose that H contains a vertex that is terminal in Γ_X. Assume that the horned graph H is $w_0 - b_1 - w_1 - b_2 - w_2$. The linear subgraph L of Γ_X with initial vertex w_2 and terminal vertex w_r is p-string of order $2(r-2) = 2p_1$ (and hence an O-string). L-prune Γ_X at the linear graph L. The resulting graph Γ' is a linear graph (with terminal white vertices) where $\Gamma' = \Gamma'(2, 1, 1, 2, 1, 2^{p_1})$. A presentation of the fundamental group of $X_{\Gamma'}$ is given by:

$$\{x_1, x_2, x_3 : x_1^2 = 1, x_1 = x_2, x_2^2 = x_3, x_3^{2^{p_1}} = 1\}.$$

This presentation is equivalent to:

$$\{x_1 : x_1^2 = 1\}.$$

Therefore $\pi_1(X) \cong \mathbb{Z}_2$ if H contains a terminal vertex of Γ_X.

We conclude that if X has finite fundamental group and the graph Γ_X is a linear graph then the lemma is true. We now show that this lemma holds for a graph Γ_X with one black vertex of degree 3 then proceed with induction for a graph Γ_X with $n > 1$ black vertices of degree 3.

Suppose that Γ_X contains one black vertex b of degree 3. The black vertex b is adjacent to the initial vertex v_1, v_2, v_3 of three terminal linear subgraphs T_1, T_2, T_3 respectively. At most one terminal linear subgraph T_1, T_2, T_3 contains a horned tree. If T_i does not contain a horned tree then T_i is a p-string. Let T_1, T_2 be p-strings. Let the terminal vertices of T_i which are terminal vertices of Γ_X be called t_i for $i = 1, 2$. Apply operation $B1$ to $st(b) \cup T_1 \cup T_2$. The resulting graph Γ' is a linear 2-stratifold. Let the associated p-string be called T'. Note that v_3 is the initial vertex of the associated p-string T' in Γ' and v_3 is not a terminal vertex of either Γ_X or Γ'. The fundamental group $\pi_1(X_{\Gamma'})$ is isomorphic to $\mathbb{Z}_{2^{k+1}}$ for $k \geq 0$ and Γ' contains a horned tree H'. Orient the graph Γ' so that vertices are ordered as $w'_0 - b'_1 - w'_1 - b'_2 - ... - b'_r - w'_r$ with corresponding edge labels $m'_1 - n'_1 - \ldots - m'_r - m'_r$. Then there is a j, where $1 < j \leq r$ such that $w'_{j-2} - b'_{j-1} - w'_{j-1} - b'_j - w'_j$ is a horned tree H'.

The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$ and by Lemma 5.1.2 if Γ' contains a horned tree H' then Γ_X contains a horned tree H. Further if $\pi_1(X_{\Gamma'})$ is isomorphic to $\mathbb{Z}_2$ then the horned tree H' of Γ' contains a terminal vertex of Γ'. It follows that $\pi_1(X_\Gamma)$ is isomorphic to $\mathbb{Z}_2$ and by Lemma 5.1.2 the horned tree H contains a terminal vertex of Γ_X.

We now show that all linear subgraphs L of Γ_X whose initial vertex v is a terminal vertex of H and whose terminal vertex w is a terminal vertex of Γ_X where $H \cap L = w$ and $v \neq w$ are O-strings. Then we show that if $\pi_1(\Gamma_X) \cong \mathbb{Z}_{2^{k+1}}$ where $k > 0$ that k corresponds to the minimal number of edges with label 2 in all O-strings L with initial vertex v and terminal vertex w.

Suppose that $\pi_1(X_{\Gamma'}) \cong \mathbb{Z}_2$. Let the horned tree H' be the subgraph $w'_0 - b'_1 - w'_1 - b'_2 - w'_2$ in Γ'. Let L' be the linear subgraph of Γ' with initial vertex w'_2 and terminal vertex w'_r. The vertex v_3 is either a nonterminal vertex of H', a terminal vertex of H', or disjoint from H'.

If v_3 is disjoint from H' in Γ' then $v_3 = w'_i$ where $2 < i < r$ and H' is properly contained in the terminal linear subgraph T_3 of Γ_X. If v_3 is a terminal vertex of H' then $v_3 = w'_2$ and the horned tree H' is the terminal linear subgraph T_3 of Γ_X. Since the linear subgraph L' is a p-string in Γ', it follows by Lemma 5.1.2, that every linear subgraph L of Γ_X whose initial vertex is w'_2 and whose terminal vertex is t_i of Γ_X is an O-string.

If v_3 is a nonterminal vertex of H' then $v_3 = w'_1$. The horned tree H contained in Γ_X contains the black vertex b. Therefore the terminal linear branches T_1, T_2, T_3 are all p-strings. T_3 is of length 2. If T_i is of length > 2 then let O_i be the linear subgraph contained in T_i whose initial vertex v is a terminal vertex of H and whose terminal vertex is a terminal vertex of Γ_X such that $O_i \cap H = v$. Then O_i is a p-string.

Suppose that $\pi_1(X_{\Gamma'}) \cong \mathbb{Z}_{2^{k+1}}$ where $k > 0$. Then H' is the subgraph of Γ' with vertices $w'_{j-2} - b'_{j-1} - w'_{j-1} - b'_j - w'_j$ where $2 < j < r$. The horned tree H' does not contain a terminal vertex of Γ'. Let L'_1 be the linear subgraph of Γ' with initial vertex w'_{j-2} and terminal vertex w'_0 and let L'_2 be the linear subgraph of Γ' with initial vertex w'_j and terminal vertex w'_r. The linear subgraphs L'_1, L'_2 are p-strings of length $2p'_1, 2p'_2$ where $p'_i \geq k$ and for at least one L'_i we have $p'_i = k$. Suppose that v_3 is contained in the linear graph whose initial vertex is w'_{j-1} and whose terminal vertex is w'_r. (If v_3 is contained in the linear graph whose initial vertex is w'_{j-1} and whose terminal vertex is w'_0 then the same argument applies.) The vertex v_3 is either a nonterminal vertex of H', a terminal vertex of H', or disjoint from H'.

If v_3 is disjoint from H' in Γ' then $v_3 = w'_i$ where $j < i < r$ and H' is properly contained in the terminal linear subgraph T_3 of Γ_X. If v_3 is a terminal vertex of H' then $v_3 = w'_j$ and H' is properly contained in the terminal linear subgraph T_3 of Γ_X. In both cases since the linear subgraph L'_2 in Γ' is a p-string, it follows by Lemma 5.1.2, that every linear subgraph L of Γ_X whose initial vertex is w'_j and whose terminal vertex t_i of Γ_X is an O-string. L'_1 is a p-string in Γ' that is disjoint from T'. By remark 5.1.1, L'_1 is contained in Γ_X. Let R_i be a linear subgraph of Γ_X whose initial vertex is w'_j and whose terminal vertex is t_i. If L'_2 contains k edges with label 2 then at least one R_i for $i = 1, 2$ contains k edges with label 2. If L'_2 does not contain k edges with label 2 then R_i contains more than k edges with label 2. Then the subgraph L'_1 of Γ' contains k edges with label 2. By remark 5.1.1, L'_1 is contained in Γ_X.

If v_3 is a nonterminal vertex of H' then $v_3 = w'_{j-1}$. The horned tree H contained in Γ_X contains the black vertex b. Therefore the terminal linear branches T_1, T_2, T_3 are all p-strings. By the same argument in the previous paragraph, all terminal p-strings T_i are of length l where $l \geq 2(k+1)$ and at least one T_i is of length $2(k+1)$.

The lemma holds for a graph Γ_X with one black vertex of degree 3. We now proceed with induction for a graph Γ_X with $n > 1$ black vertices of degree 3.

Suppose that Γ_X contains $n > 1$ black vertices of degree 3. Let b be a black vertex of degree 3 that is adjacent to the vertices v_1, v_2, v_3 such that v_i is the initial vertex of a terminal linear subgraph T_i for $i = 1, 2$. (The black vertex b is an outermost such vertex, in that at least two components of

$\Gamma_X \setminus st(b)$ contains only vertices with degree < 3.) If T_i does not contain a horned tree then T_i is a p-string. If either T_1 or T_2 contains a horned tree, then by corollary 4.2.3, there exists another outermost black vertex b' of degree 3 that is adjacent to the initial vertices of two terminal linear branches T_1', T_2'. Since X_Γ has finite fundamental group the two terminal linear branches T_1', T_2' do not contain a horned tree. We assume T_1 and T_2 do not contain a horned tree. Then T_1 and T_2 are terminal p-strings. Let the terminal vertices of T_i which are terminal vertices of Γ_X be called t_i for $i = 1, 2$. Apply operation $B1$ to $st(b) \cup T_1 \cup T_2$. The resulting graph Γ' has $n - 1$ black vertices of degree 3. Let the associated p-string be called T' and let the terminal vertex of T' and Γ' be called t'. By the induction hypothesis, $\pi_1(X_{\Gamma'})$ is isomorphic to $\mathbb{Z}_{2^{k+1}}$ for $k \geq 0$ and Γ' contains a horned tree H'.

The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$ and by Lemma 5.1.2 if Γ' contains a horned tree H' then Γ_X contains a horned tree H. Further if $\pi_1(X_{\Gamma'})$ is isomorphic to $\mathbb{Z}_2$ then the horned tree H' of Γ' contains a terminal vertex of Γ'. By Lemma 5.1.2, this implies that $\pi_1(X_\Gamma)$ is isomorphic to $\mathbb{Z}_2$ and the horned tree H contains a terminal vertex of Γ_X.

Let L' be a linear subgraph of Γ' whose initial vertex v' is a terminal vertex of H' and whose terminal vertex w' is a terminal vertex of Γ' where $L' \cap H' = v'$ and $w' \neq v'$. By the induction hypothesis L' is an O-string. By remark 5.1.1 and lemma 5.1.2, if L' is disjoint from $T' \setminus v_3$ then L' is an O-string in Γ_X that is disjoint from $st(b) \cup T_1 \cup T_2$ and the initial vertex v' of L' is a terminal vertex of H_T. We assume L' is not disjoint from $T' \setminus v_3$. Then the terminal vertex w' of L' is t' which is the terminal vertex of T'. The vertex v_3 is either a nonterminal vertex of H', a terminal vertex of H', or disjoint from H'.

If v_3 is disjoint from H' then L' properly contains the p-string T'. If v_3 is a terminal vertex of H' then L' is the p-string T'. In both cases H' is contained in Γ_X. It follows by Lemma 5.1.2, that every linear subgraph L of Γ_X whose initial vertex is v' and whose terminal vertex is t_i of Γ_X is an O-string.

If v_3 is a nonterminal vertex of H' then L' is properly contained in T'. For the terminal p-strings T_i of Γ_X, order the vertices $w_0^i - b_1^i - w_1^i - b_2^i - ... - b_{r_i}^i - w_{r_i}^i$ where $w_0^i = v_i$ and $w_{r_i}^i = t_i$ of Γ_X for $i = 1, 2$. The terminal linear subgraphs T_1, T_2 of Γ_X intersect the horned tree H at the subgraphs $w_0^i - b_1^i - w_1^i$. The terminal linear subgraphs of T_1, T_2 whose initial vertex is w_1^i and whose terminal vertex is t_i is an O-string.

Suppose that $\pi_1(X_\Gamma) \cong \mathbb{Z}_{2^{k+1}}$ where $k > 0$. Then H' does not contain a terminal vertex of Γ'. By the induction hypothesis, there exists an O-string L' of Γ' whose initial vertex is a terminal vertex v' of H' and whose terminal vertex w' is a terminal vertex of Γ' where $L' \cap H' = v'$ and L' contains k edges with label 2. The number k is minimal among all such O-strings. By remark 5.1.1 and lemma 5.1.2, if L' is disjoint from $T' \setminus v_3$ then L' is an O-string in Γ_X that is disjoint from $st(b) \cup T_1 \cup T_2$ and the initial vertex v' of L' is a terminal vertex of H. We assume L' is not disjoint from $T' \setminus v_3$.

The vertex v_3 is either a nonterminal vertex of H', a terminal vertex of H', or disjoint from H'. If v_3 is disjoint from H' then L' properly contains the p-string T'. If v_3 is a terminal vertex of H' then L' is the p-string T'. In both cases H' is contained in Γ_X. It follows by Lemma 5.1.2, that at least one linear subgraph L of Γ_X whose initial vertex is v' and whose terminal vertex is t_i of Γ_X is an O-string with k edges with label 2.

If v_3 is a nonterminal vertex of H' then L' is properly contained in T'. The terminal linear subgraph T' contains $k+1$ edges with label 2. For the terminal p-strings T_i of Γ_X, order the vertices $w_0^i - b_1^i - w_1^i - b_2^i - ... - b_{r_i}^i - w_{r_i}^i$ where $w_0^i = v_i$ and $w_{r_i}^i = t_i$ of Γ_X for $i = 1, 2$. The terminal linear subgraphs T_1, T_2 of Γ_X intersect the horned tree H at the subgraphs $w_0^i - b_1^i - w_1^i$ and by lemma 5.1.2 at least one of the terminal linear subgraph T_1, T_2 contains $k+1$ edges with label 2. Therefore at least one of the terminal linear subgraphs of T_1, T_2 whose initial vertex is w_1^i and whose terminal vertex is t_i is an O-string with k edges with label 2.

$\square$

In the proof of the previous lemma, corollary 4.2.3 insured us that we could find an outermost black vertex of degree 3 that is adjacent to the initial vertices of terminal p-strings for Γ_X with $n > 1$ black vertices of degree 3. A similar statement to corollary 4.2.3 is now made for Γ_X containing a black terminal vertex. This statement follows from Lemma 4.2.2.

Corollary 5.1.4. *Let X be a trivalent 2-stratifold. If Γ_X is a tree that contains $n > 1$ black vertices of degree 3, all white vertices are of degree ≤ 2, and one black terminal vertex then Γ_X contains at least two black vertices of degree 3 that are adjacent to the initial vertex of two terminal linear subgraphs.*

Lemma 5.1.5. *Let X be a pruned trivalent 2-stratifold where Γ_X has a label 2 for all edges incident to a terminal white vertex of genus* 0*. Let Γ_X have one white terminal vertex of genus -1 with incident edge label* 1 *while all other white vertices are genus* 0*, all terminal vertices are white, and all white vertices are of degree ≤ 2. If $\pi_1(X)$ is finite then all of the following hold:*

1. *Let L be a linear subgraph of Γ_X whose initial vertex v is the white vertex of genus -1 and whose terminal vertex w is a terminal vertex of Γ_X where $w \neq v$. Then L is an O-string.*

2. *The fundamental group $\pi_1(X)$ is isomorphic to $\mathbb{Z}_{2^{k+1}}$ where the integer $k > 0$ corresponds to the minimal number of edges with label 2 in all L whose initial vertex v is the white vertex of genus -1 and whose terminal vertex w is a terminal vertex of Γ_X where $w \neq v$.*

Proof. By theorem 4.3.7 the fundamental group $\pi_1(X)$ is finite implies Γ_X is a tree that contains no horned trees. Let v be the terminal white vertex of genus -1.

Suppose that Γ_X has no black vertices of degree 3. The graph Γ_X is a linear graph. Orient the graph Γ_X so that vertices are ordered as $w_0 - b_1 - w_1 - b_2 - ... - b_r - w_r$ with corresponding edge labels $m_1 - n_1 - \ldots - m_r - n_r$ and $w_0 = v$. By assumption the labels $m_1 = 1, n_1 = 2$ and $m_r = 1, n_r = 2$. If there exists a subgraph $w_{i-1} - b_i - w_i$ for $1 < i < r$ with successive labels $m_i = 2, n_i = 1$ then Γ_X contains a horned tree. Therefore each subgraph $w_{i-1} - b_i - w_i$ for $1 < i < r$ has successive labels $m_i = 1, n_i = 2$. The graph Γ_X is an O-string. L-prune Γ_X, the resulting graph Γ' is a linear graph with vertices $w_0 - b'_1 - w'_1$ where $\Gamma' = \Gamma'(1, 2^r)$ and w_0 has genus -1. A presentation of the fundamental group of $X_{\Gamma'}$ is given by:

$$\{x_1, y, c : x_1^{2^r} = 1, x_1 = c, cy^2 = 1\}.$$

This presentation is equivalent to:

$$\{y : y^{2^{r+1}} = 1\}.$$

Then $\pi_1(X) \cong \mathbb{Z}_{2^{r+1}}$ where r is the number of edges with label 2 in the O-string Γ_X.

Suppose that Γ_X contains one black vertex b of degree 3. The black vertex b is adjacent to the initial vertex v_1, v_2, v_3 of three terminal linear subgraphs T_1, T_2, T_3 respectively. One terminal linear subgraph T_1, T_2, T_3 contains the vertex v. If T_i does not contain v then T_i is a p-string. Let T_1, T_2 be p-strings. Let the terminal vertices of T_1, T_2 which are terminal vertices of Γ_X be called t_i.

Apply operation $B1$ to $st(b) \cup T_1 \cup T_2$. The resulting graph Γ' is linear graph. Orient the graph Γ' so that vertices are ordered as $w'_0 - b'_1 - w'_1 - b'_2 - ... - b'_r - w'_r$ with corresponding edge labels $m'_1 - n'_1 - \ldots - m'_r - n'_r$ and let $w'_0 = v$. Then each subgraph $w'_{i-1} - b'_i - w'_i$ for $1 \leq i \leq r$ has successive labels $m'_i = 1, n'_i = 2$ and $\pi_1(X) \cong \mathbb{Z}_{2^{r+1}}$. The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$. By Lemma 5.1.2, If $v_3 = w'_i$ for $0 \leq i < r$ then the linear subgraph L_i in Γ_X with initial vertex $w'_0 = v$ and terminal vertex t_i is an O-string and at least one L_i contains r edges with label 2.

Suppose that Γ_X contains $n > 1$ black vertices of degree 3. Let b be a black vertex of degree 3 that is adjacent to the vertices v_1, v_2, v_3 such that v_i is the initial vertex of a terminal linear subgraph T_i for $i = 1, 2$. (The black vertex b is an outermost such vertex, in that at least two components of $\Gamma_X \setminus st(b)$ contains only vertices with degree < 3.) If T_i does not contain v then T_i is p-string. If T_i contains v then by corollary 4.2.3, there exists another outermost black vertex b' of degree 3 that is adjacent to the initial vertex of two terminal linear branches T'_1, T'_2. Then T'_1 and T'_2 are terminal p-strings. We assume that both T_1 and T_2 are terminal p-strings. Let the terminal vertices of T_i which are terminal vertices of Γ_X be called t_i for $i = 1, 2$. Apply operation $B1$ to $st(b) \cup T_1 \cup T_2$. The resulting graph Γ' has $n - 1$ black vertices of degree 3. Let the associated p-string be called T' and let the terminal vertex of T' and Γ' be called t'.

By the induction hypothesis, $\pi_1(X_{\Gamma'})$ is isomorphic to $\mathbb{Z}_{2^{k+1}}$ for $k > 0$. The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$.

Let L' be a linear subgraph of Γ' whose initial vertex is v and whose terminal vertex w' is a terminal vertex of Γ' where $v \neq w'$. By the induction hypothesis L' is an O-string. If L' is disjoint from $T' \setminus v_3$ then L' is disjoint from T'. By remark 5.1.1, L' is an O-string in Γ_X that is disjoint from $v_3 \cup st(b) \cup T_1 \cup T_2$. We assume L' is not disjoint from $T' \setminus v_3$. Then the terminal vertex w' of L' is t' which is the terminal vertex of T'. By Lemma 5.1.2 it follows that every linear subgraph L of Γ_X whose initial vertex is v and whose terminal vertex is t_i of Γ_X is an O-string.

By the induction hypothesis, there exists an O-string L' of Γ' whose initial vertex is v and whose terminal vertex w' is a terminal vertex of Γ' where $v \neq w'$ and L' contains $k > 0$ edges with label 2. The number k is minimal among all such O-strings. If L' is disjoint from $T' \setminus v_3$ then L' is disjoint from T'. By remark 5.1.1, L' is an O-string in Γ_X that is disjoint from $v_3 \cup st(b) \cup T_1 \cup T_2$. We assume L' is not disjoint from $T' \setminus v_3$. Then the terminal vertex w' of L' is t' which is the terminal vertex of T'. By Lemma 5.1.2 there exists an O-string of Γ_X whose initial vertex is v and whose terminal vertex is t_i of Γ_X with exactly k edges with label 2 for some $i = 1, 2$.

□

Lemma 5.1.6. *Let X be a pruned trivalent 2-stratifold where Γ_X has a label 2 for all edges incident to a terminal white vertex of genus 0. Let Γ_X have all white vertices of genus 0, one black terminal vertex, and all white vertices are of degree ≤ 2. If $\pi_1(X)$ is finite then all of the following hold:*

1. *Let L be a linear subgraph of Γ_X whose initial vertex v is the white vertex adjacent to the black terminal vertex and whose terminal vertex w is a white terminal vertex of Γ_X. Then L is an O-string.*

2. *The fundamental group $\pi_1(X)$ is isomorphic to $\mathbb{Z}_{3(2^k)}$ where the integer $k > 0$ corresponds to the minimal number of edges with label 2 in all L whose initial vertex v is the white vertex adjacent to the black terminal vertex and whose terminal vertex w is a white terminal vertex of Γ_X.*

Proof. By theorem 4.3.7 the fundamental group $\pi_1(X)$ is finite implies that Γ_X is a tree that contains no horned trees. Let b'' be the black terminal vertex of Γ_X and let v be the white vertex adjacent to b''.

Suppose that Γ_X has no black vertices of degree 3. The graph Γ_X is a linear graph. Orient the graph Γ_X so that vertices are ordered as $b_1 - w_1 - b_2 - ... - b_{r+1} - w_{r+1}$ with corresponding edge labels $n_1 - \ldots - m_{r+1} - n_{r+1}$ where $b_1 = b''$. By assumption the labels $m_r = 1, n_r = 2$. If there exists a subgraph $w_{i-1} - b_i - w_i$ for $1 < i < r + 1$ with successive labels $m_i = 2, n_i = 1$ then Γ_X contains a horned tree. Therefore each subgraph $w_{i-1} - b_i - w_i$ for $1 < i < r + 1$ has successive labels $m_i = 1, n_i = 2$. The linear graph L with initial vertex w_1 and terminal vertex w_{r+1} in Γ_X is an O-string. L-prune Γ_X, the resulting graph Γ' has vertices $b_1 - w_1'$ with edge label $n = 3 * 2^r$. A presentation of the fundamental group of $X_{\Gamma'}$ is given by:

$$\{x_1 : x_1^{3*2^r} = 1\}.$$

Then $\pi_1(X) \cong \mathbb{Z}_{3*2^r}$ where r is the number of edges with label 2 in the O-string L.

Suppose that Γ_X contains one black vertex b of degree 3. The black vertex b is adjacent to the initial vertex v_1, v_2, v_3 of three terminal linear subgraphs T_1, T_2, T_3 respectively. One terminal linear subgraph T_1, T_2, T_3 contains the black terminal vertex b'' of Γ_X. If T_i does not contain b'' then T_i is a p-string. Let T_1, T_2 be p-strings. Let the terminal vertices of T_1, T_2 which are terminal vertices of Γ_X be called t_i. Apply operation $B1$ to $st(b) \cup T_1 \cup T_2$. The resulting graph Γ' is linear graph. Orient the graph Γ' so that vertices are ordered as $b'_1 - w'_1 - b'_2 - ... - b'_{r+1} - w'_{r+1}$ with corresponding edge labels $n'_1 - \ldots - m'_{r+1} - n'_{r+1}$. Then each subgraph $w'_{i-1} - b'_i - w'_i$ for $1 \leq i \leq r+1$ has successive labels $m'_i = 1, n'_i = 2$ and $\pi_1(X) \cong \mathbb{Z}_{3*2^r}$. The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$. By Lemma 5.1.2, If $v_3 = w'_i$ for $1 \leq i < r+1$ then the linear subgraph L_i in Γ_X with initial vertex $w'_0 = v$ and terminal vertex t_i is an O-string and at least one L_i contains r edges with label 2.

Suppose that Γ_X contains $n > 1$ black vertices of degree 3. Let b be a black vertex of degree 3 that is adjacent to the vertices v_1, v_2, v_3 such that v_i is the initial vertex of a terminal linear subgraph T_i for $i = 1, 2$. (The black vertex b is an outermost such vertex, in that at least two components of $\Gamma_X \setminus st(b)$ contains only vertices with degree < 3.) By corollary 5.1.4, If T_i contains b'' then there exists another outermost black vertex b' of degree 3 that is adjacent to the initial vertex of two terminal linear branches T'_1, T'_2 which do not contain b''. We then assume that both T_1, T_2 do not contain b'' and T_1, T_2 are terminal p-strings. Let the terminal vertices of T_i which are terminal vertices of Γ_X be called t_i for $i = 1, 2$. Apply operation $B1$ to $st(b) \cup T_1 \cup T_2$. The resulting graph Γ' has $n - 1$ black vertices of degree 3. Let the associated p-string be called T' and let the terminal vertex of T' and Γ' be called t'.

By the induction hypothesis, $\pi_1(X_{\Gamma'})$ is isomorphic to $\mathbb{Z}_{3*2^k}$ for $k > 0$. The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$.

Let L' be a linear subgraph of Γ' whose initial vertex is v and whose terminal vertex w' is a white terminal vertex of Γ'. By the induction hypothesis L' is an O-string. If L' is disjoint from $T' \setminus v_3$ then L' is disjoint from T'. By remark 5.1.1, L' is an O-string in Γ_X that is disjoint from $v_3 \cup st(b) \cup T_1 \cup T_2$. We assume L' is not disjoint from $T' \setminus v_3$. Then the terminal vertex w' of L' is t' which is the terminal vertex of T'. By Lemma 5.1.2 it follows that every linear subgraph L of Γ_X whose initial vertex is v and whose terminal vertex is t_i of Γ_X is an O-string.

By the induction hypothesis, there exists an O-string L' of Γ' whose initial vertex is v and whose terminal vertex w' is a white terminal vertex of Γ' and L' contains $k > 0$ edges with label 2. The number k is minimal among all such O-strings. If L' is disjoint from $T' \setminus v_3$ then L' is disjoint from T'. By remark 5.1.1, L' is an O-string in Γ_X that is disjoint from $v_3 \cup st(b) \cup T_1 \cup T_2$. We assume L' is not disjoint from $T' \setminus v_3$. Then the terminal vertex w' of L' is t' which is the terminal vertex of T'.

By Lemma 5.1.2 there exists an O-string of Γ_X whose initial vertex is v and whose terminal vertex is t_i of Γ_X with exactly k edges with label 2 for some $i = 1, 2$.

□

The dihedral group of order $2n$ will be denoted by D_n.

Lemma 5.1.7. *Let X be a pruned trivalent 2-stratifold where Γ_X has a label 2 for all edges incident to a terminal white vertex of genus* 0. *Let Γ_X have all white vertices of genus* 0, *all terminal vertices are white, and there is exactly one white vertex v'' of degree 3 while all other white vertices are of degree ≤ 2. Let e_i be the edges incident to v'' for $1 \leq i \leq 3$. Let L^i be a linear subgraph of Γ_X whose initial vertex is v'', whose terminal vertex w is a terminal vertex of Γ_X, and L^i contains e_i. If $\pi_1(X)$ is finite then all of the following hold:*

1. *The linear subgraph L^i is an O-string.*
2. *There exists an L^i for $i = 1, 2$ of Γ_X that contains only one edge labelled with 2.*
3. *The fundamental group $\pi_1(X)$ is isomorphic to D_{2^k}, where the integer $k > 0$ corresponds to the minimal number of edges with label 2 in all L^3 of Γ_X.*

Proof. By theorem 4.3.7 the fundamental group $\pi_1(X)$ is finite implies that Γ_X is a tree that contains neither a horned tree disjoint from v'' nor a horned tree with v'' as a terminal vertex.

Suppose that Γ_X has no black vertices of degree 3. Define L_i to be the linear subgraph whose initial vertex is v'', whose terminal vertex is a terminal vertex of Γ_X, and L_i contains the edge e_i. If at least one L_i contains a horned tree then X_Γ has infinite fundamental group. Then each L_i is a p-string of length $2p_i$. The 2-stratifold X_Γ has fundamental group that can be represented with the following presentation:

$$\{c_1, c_2, c_3 : c_1^{2^{p_1}} = 1, c_2^{2^{p_2}} = 1, c_3^{2^{p_3}} = 1, c_1c_2c_3 = 1\}.$$

The presentation is an F-group. Each $p_i > 0$ and so the presentation is a finite non-cyclic F-group. Therefore (without a loss of generality) $p_1 = 1$, $p_2 = 1$, and $p_3 \geq 1$ and $\pi_1(X_\Gamma)$ is the dihedral group $D_{2^{p_3}}$. It follows that L_1, L_2 are p-strings of length 2 and L_3 is a p-string of length $2p_3$.

Suppose that Γ_X contains $n > 0$ black vertices of degree 3. Let b be a black vertex of degree 3 that is adjacent to the vertices v_1, v_2, v_3 such that v_i is the initial vertex of a terminal linear subgraph

T_i for $i = 1, 2$. (The black vertex b is an outermost such vertex, in that at least two components of $\Gamma_X \setminus st(b)$ contains only vertices with degree < 3.) If at least one T_i contains a horned tree then X_Γ has infinite fundamental group. Then T_1 and T_2 are terminal p-strings. Let the terminal vertices of T_i which are terminal vertices of Γ_X be called t_i for $i = 1, 2$. Apply operation $B1$ to $st(b) \cup T_1 \cup T_2$. The resulting graph Γ' has $n - 1$ black vertices of degree 3. Let the associated p-string be called T' and let the terminal vertex of T' and Γ' be called t'.

By the induction hypothesis, $\pi_1(X_{\Gamma'})$ is isomorphic to D_{2^k} for $k > 0$. The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$.

Let L' be a linear subgraph of Γ' whose initial vertex is v'' and whose terminal vertex w' is a terminal vertex of Γ'. By the induction hypothesis L' is an O-string. If L' is disjoint from $T' \setminus v_3$ then L' is disjoint from T'. By remark 5.1.1, L' is an O-string in Γ_X that is disjoint from $v_3 \cup st(b) \cup T_1 \cup T_2$. We assume L' is not disjoint from $T' \setminus v_3$. Then the terminal vertex w' of L' is t' which is the terminal vertex of T'. By Lemma 5.1.2 it follows that every linear subgraph L of Γ_X whose initial vertex is v'' and whose terminal vertex is t_i of Γ_X is an O-string.

By the induction hypothesis, there exists an O-string L'_i whose initial vertex is v'' and whose terminal vertex is a terminal vertex of Γ' with exactly p_i edges with label 2 where $p_i = 1$ if $i = 1, 2$ and $p_i \geq 1$ if $i = 3$. If L'_i is disjoint from $T' \setminus v_3$ then L'_i is disjoint from T'. By remark 5.1.1, L'_i is an O-string in Γ_X that is disjoint from $v_3 \cup st(b) \cup T_1 \cup T_2$. We assume L'_i is not disjoint from $T' \setminus v_3$. Then the terminal vertex w' of L' is t' which is the terminal vertex of T'. By Lemma 5.1.2 there exists an O-string of Γ_X whose initial vertex is v'' and whose terminal vertex is t_i of Γ_X with exactly p_i edges with label 2 for some $i = 1, 2$.

□

5.2 Trivalent 2-stratifolds with finite fundamental group

For a trivalent bicolored graph Γ, we now describe the necessary and sufficient conditions on Γ for $\pi_1(X_\Gamma)$ to be finite where $\Gamma = \Gamma_X$.

In this section, we assume that Γ is a tree that satisfies one of the following conditions:

1. *The graph Γ has exactly one black terminal vertex and all white vertices are genus 0.*
2. *The graph Γ has exactly one white vertex of genus -1 while all other white vertices are genus 0 and all terminal vertices are white.*

3. *The graph* Γ *contains all white vertices of genus* 0 *and all terminal vertices are white.*

These are necessary conditions by lemma 2.3.3 for X_Γ to have finite fundamental group. A 2-stratifold X_Γ with a graph Γ that contains a vertex of genus -1 or a black terminal vertex is never 1-connected. For graphs Γ with all white terminal vertices and all white vertices of genus 0, the associated 2-stratifold X_Γ can be 1-connected. **Throughout this section we assume that X_Γ is not 1-connected and X_Γ is pruned.**

Core-reduced graphs were defined in [10] for trivalent graphs Γ that are homotopically equivalent to S^1. In summary nontrivial core-reduced graphs Γ_C are pruned subgraphs of Γ_X that carry the fundamental group information of X_Γ. We adapt the definition for when Γ is a tree. An efficient algorithm to decide whether or not a trivalent 2-stratifold is 1-connected was given in [5]. We will implicitly use this algorithm in our definition of a core reduced graph.

A vertex of Γ with degree > 2 will be called a **branch vertex**. Let b_0 be a black branch vertex of distance 1 from a terminal vertex w_0 and let C_1, C_2 be subgraphs of Γ corresponding to the components of $\Gamma \setminus (st(b_0) \cup w_0)$. Then such a black branch vertex b_0 is an called **outermost** if at least one C_i contains no black branch vertices distance 1 to a terminal vertex. We refer to a labelled graph Γ as 1-connected if X_Γ is 1-connected.

If the graph Γ does not contain a black branch vertex of distance 1 to a terminal vertex then Γ is core-reduced. If Γ contains a black branch vertex of distance 1 to a terminal vertex we let $B = \{b_{01}, \ldots, b_{0k}\}$ be the set of all outermost black branch vertices where each b_{0i} has distance 1 from a terminal vertex w_{0i}. Choose a component of $\Gamma \setminus (st(b_{0i}) \cup w_{0i})$ corresponding to a subgraph C_i of Γ that does not contain a black branch vertex of distance 1 to a terminal vertex to be denoted T_{0i}. If there exists at least two components T_{0i} that are not 1-connected let $\Gamma_0 = \emptyset$. If one component T_{0i} is not 1-connected and $\Gamma \setminus (T_{0i} \cup st(b_{0i}) \cup w_{0i})$ is not 1-connected then let $\Gamma_0 = \emptyset$. If each T_{0i} is 1-connected and $\Gamma \setminus (T_{0i} \cup st(b_{0i}) \cup w_{0i})$ is not 1-connected then let $\Gamma_0' = \Gamma \setminus (\bigcup st(b_{0i}) \cup \bigcup w_{0i} \cup \bigcup T_{0i})$. If exactly one component T_{0i} is not 1-connected and $\Gamma \setminus (T_{0i} \cup (st(b_{0i}) \cup w_{0i})$ is 1-connected then let $\Gamma_0' = T_{0i}$. If Γ_0' is pruned then let $\Gamma_0 = \Gamma_0'$, otherwise let Γ_0 be the pruned Γ_0'. For $\Gamma_0 \neq \emptyset$, we have that $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_0})$ since $r^{-1}(b_{i0})$ is contractible in X_Γ. For $\Gamma_0 = \emptyset$, we have that $\pi_1(X_\Gamma)$ is infinite.

By induction, If Γ_{n-1} contains a black branch vertex of distance 1 to a terminal vertex we let $B_{n-1} = \{b_{n-1,1}, \ldots, b_{n-1,k_{n-1}}\}$ be the set of all outermost black branch vertices where each $b_{n-1,i}$

has distance 1 from a terminal vertex $w_{n-1,i}$. Choose a component of $\Gamma_{n-1} \setminus (st(b_{n-1,i}) \cup w_{n-1,i})$ corresponding to a subgraph C_i of Γ_{n-1} that does not contain a black branch vertex of distance 1 to a terminal vertex to be denoted $T_{n-1,i}$. If there exists at least two components $T_{n-1,i}$ that are not 1-connected let $\Gamma_n = \emptyset$. If one component $T_{n-1,i}$ is not 1-connected and $\Gamma \setminus (T_{n-1,i} \cup st(b_{n-1,i}) \cup w_{n-1,i})$ is not 1-connected then let $\Gamma_n = \emptyset$. If each $T_{n-1,i}$ is 1-connected and $\Gamma \setminus (T_{n-1,i} \cup st(b_{n-1,i}) \cup w_{n-1,i})$ is not 1-connected then let $\Gamma'_n = \Gamma_{n-1} \setminus (\bigcup st(b_{n-1,i}) \cup \bigcup w_{n-1,i} \cup \bigcup T_{n-1,i})$. If exactly one component $T_{n-1,i}$ is not 1-connected and $\Gamma_{n-1} \setminus (T_{n-1,i} \cup st(b_{n-1,i}) \cup w_{n-1,i})$ is 1-connected then let $\Gamma'_n = T_{n-1,i}$. If Γ'_n is pruned the let $\Gamma_n = \Gamma'_n$, otherwise let Γ_n be the pruned Γ'_n.

We define our **core reduced graph** Γ_C of Γ as follows:

$$\Gamma_C = \begin{cases} \emptyset, & \text{if } \Gamma_n = \emptyset \text{ for some } n \geq 0, \text{otherwise} \\ \Gamma_n, & \text{for the smallest } n \text{ such that } \Gamma_n \text{ does not contain a black branch vertex of} \\ & \text{distance 1 to a terminal vertex} \end{cases}$$

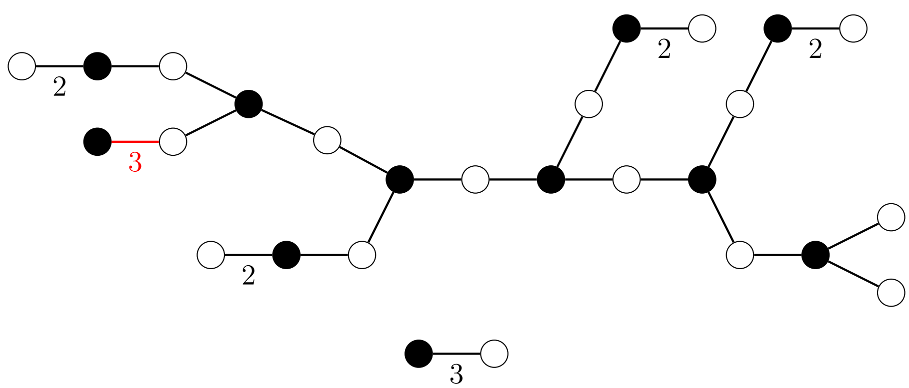


Figure 5.3: A trivalent graph Γ and its core reduced graph Γ_C. The core reduced graph is composed of the red edge along with the incident vertices.

For a core reduced graph Γ_C of Γ where $\Gamma_C \neq \emptyset$, we have that $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$. While if $\Gamma_C = \emptyset$ then $\pi_1(X_\Gamma)$ is infinite.

A **pseudo-projective plane of order** $k > 2$ is a 2-stratifold that is obtained by attaching a 2-cell to a circle by the map $z \to z^k$. A pseudo-projective plane of order 3 is a trivalent 2-stratifold. A model of such a space can be seen in figure 5.4. The bipartite labelled graph of a pseudo-projective plane of order 3 is the core reduced graph seen in figure 5.3.

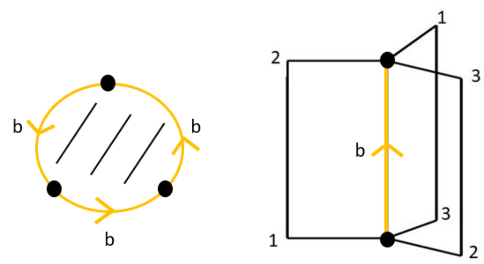


Figure 5.4: A pseudo-projective plane of order 3 obtained by identifying the arcs on the boundary of a disk and a regular neighborhood of the singular curve.

Corollary 5.2.1. *Let Γ be a bicolored pruned trivalent graph such that X_Γ is a trivalent 2-stratifold that has finite (nontrivial) fundamental group. Let Γ_C be the core reduced graph of Γ. Then Γ is one of the cases below:*

1. *Γ has exactly one black terminal vertex and all white vertices are genus 0. Then the graph Γ_C contains exactly one black terminal vertex, all white vertices are genus 0, and either all edges of Γ_C incident to a terminal white vertex have label 2 or X_{Γ_C} is a pseudo-projective plane of order 3.*
2. *Γ has exactly one white vertex of genus -1 while all other white vertices are genus 0 and all terminal vertices are white. Then the graph Γ_C either contains one white vertex of genus -1 while all other white vertices are genus 0, all terminal vertices are white, and all edges of Γ_C incident to a terminal white vertex of genus 0 have label 2 or X_{Γ_C} is a projective plane.*
3. *Γ has all white terminal vertices and white vertices are of genus 0. Then the graph Γ_C contains all white vertices of genus 0, all terminal vertices are white, and all edges of Γ_C incident to a terminal vertex have label 2.*

Proof. The graph Γ_C is a pruned subgraph of Γ. Since $\pi(X_\Gamma)$ is finite, $\Gamma_C \neq \emptyset$.

(1.) Γ_C contains at most one black terminal vertex and all white vertices are of genus 0. Suppose that Γ_C does not contain a black terminal vertex. If Γ is not 1-connected then Γ_C is not 1-connected. Let Γ_0 be the subgraph of Γ corresponding to Γ_C. Attach to each black vertex that is not the terminal black vertex and is not contained in the subgraph Γ_0 of Γ a white vertex of genus 0 with edge label 1. Then there is an epimorphism from $\pi_1(X_\Gamma) \to \mathbb{Z}_3 \star \pi_1(X_{\Gamma_C})$. The graph Γ_C contains a black terminal vertex.

The graph Γ_C contains no terminal q-strings and no black branch vertex of distance 1 to a terminal vertex. Let v be a white terminal vertex of Γ_C. If v is not contained in a terminal p-string then v is adjacent to the black terminal vertex and X_{Γ_C} is a pseudo-projective plane of order 3. Otherwise v is contained in a terminal p-string and the edge label incident to v is 2.

(2.) Γ_C contains at most one white vertex of genus -1 while all other vertices are genus 0 and all terminal vertices are white. Suppose that Γ_C does not contain a white vertex of genus -1. If Γ is not 1-connected then Γ_C is not 1-connected. Let Γ_0 be the subgraph of Γ corresponding to Γ_C. Attach to each black vertex not contained in the subgraph Γ_0 of Γ a white vertex of genus 0 with edge label 1. Then there is an epimorphism from $\pi_1(X_\Gamma) \to \mathbb{Z}_2 \star \pi_1(X_{\Gamma_C})$. The graph Γ_C contains the white vertex of genus -1.

The graph Γ_C contains no terminal q-strings and no black branch vertex of distance 1 to a terminal vertex. If Γ_C contains a white terminal vertex v of genus 0 then v is contained in a terminal p-string and the edge label incident to v is 2. If Γ_C contains no white terminal vertices of genus 0 then X_{Γ_C} is a projective plane.

(3.) Γ_C contains all white terminal vertices and all white vertices are of genus 0. The graph Γ_C contains no terminal q-strings and no black branch vertex of distance 1 to a terminal vertex. If v is a white terminal vertex of genus 0 then the incident edge label is 2.

□

We determine the finite 2-stratifold groups as this will simplify our classification results.

Theorem 5.2.2. *Let Γ be a bicolored pruned trivalent graph. If X_Γ has finite fundamental group then $\pi_1(X_\Gamma)$ is isomorphic to either $\mathbb{Z}_{2^{k+1}}$, $\mathbb{Z}_{3*2^k}$, $D_{2^{k+1}}$ where $k \geq 0$.*

Proof. Let Γ_C be the core reduced graph of Γ.

Suppose that Γ has exactly one black terminal vertex and all white vertices are genus 0. By corollary 5.2.1, the graph Γ_C contains exactly one black terminal vertex, all white vertices are genus 0, and either all edges of Γ_C incident to a terminal white vertex have label 2 or X_{Γ_C} is a pseudo-projective plane of order 3. If X_{Γ_C} is a pseudo-projective plane of order 3 then $\pi_1(X_\Gamma) \cong \mathbb{Z}_3$. Otherwise by theorem 4.3.7, Γ_C has all white vertices of degree ≤ 2, and contains no horned tree. Let L be a linear subgraph of Γ_C whose initial vertex v is the white vertex adjacent to the black terminal vertex and whose terminal vertex w is a white terminal vertex of Γ_C. Then by lemma

5.1.6, L is an O-string, $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_{3*2^k}$ where $k > 0$, and the integer k corresponds to the minimal number of edges with label 2 in all L whose initial vertex v is the white vertex adjacent to the black terminal vertex and whose terminal vertex w is a white terminal vertex of Γ_C.

Suppose that Γ has exactly one white vertex of genus -1 while all other white vertices are genus 0 and all terminal vertices are white. By corollary 5.2.1, the graph Γ_C either contains one white vertex of genus -1 while all other white vertices are genus 0, all terminal vertices are white, and all edges of Γ_C incident to a terminal white vertex of genus 0 have label 2 or X_{Γ_C} is a projective plane. If X_{Γ_C} is a projective plane then $\pi_1(X_\Gamma) \cong \mathbb{Z}_2$. Otherwise by theorem 4.3.7, the white vertex of genus -1 of Γ_C is terminal and has incident edge label 1, Γ_C contains all white vertices of degree ≤ 2, and Γ_C contains no horned tree. Let L be a linear subgraph of Γ_C whose initial vertex v is the white vertex of genus -1 and whose terminal vertex w is a white terminal vertex of Γ_C where $w \neq v$. Then by lemma 5.1.5, L is an O-string, $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_{2^k}$ where $k > 1$, and the integer k corresponds to the minimal number of edges with label 2 in all L whose initial vertex v is the white vertex of genus -1 and whose terminal vertex w is a white terminal vertex of Γ_C.

Suppose that Γ contains all white vertices of genus 0 and all terminal vertices are white. By corollary 5.2.1, Γ_C contains all white vertices of genus 0, all terminal vertices are white and all edges of Γ_C incident to a terminal white vertex has label 2. By theorem 4.3.7, either Γ_C has all white vertices of degree ≤ 2 and contains at most one horned tree or Γ_C has exactly one white vertex v'' of degree 3 while all other white vertices are of degree ≤ 2 and contains no horned tree H_T such that either v'' and H_T are disjoint or v'' is a terminal vertex of H_T. We now look at these two cases.

Suppose that Γ_C has all white vertices of degree ≤ 2 and contains at most one horned tree. By lemma 5.1.3, Γ_C contains a horned tree H_T and if L is a linear subgraph of Γ_C whose initial vertex v is a terminal vertex of H_T and whose terminal vertex w is a white terminal vertex of Γ_C where $L \cap H_T = v$ and $w \neq v$ then L is an O-string. Further by lemma 5.1.3, $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_{2^{k+1}}$ where the integer $k = 0$ if H_T contains a terminal vertex of Γ_X and $k > 0$ otherwise. The integer $k > 0$ corresponds to the minimal number of edges with label 2 in all linear subgraphs L whose initial vertex v is a terminal vertex of H_T and whose terminal vertex w is a terminal vertex of Γ_X where $L \cap H_T = v$ and $w \neq v$.

Suppose that Γ_C has exactly one white vertex v'' of degree 3 while all other white vertices are of degree < 3, and contains no horned tree H_T such that either v'' and H_T are disjoint or v'' is a terminal vertex of H_T. Let e_i be the edges incident to v'' for $1 \leq i \leq 3$. Let L^i be a linear subgraph of Γ_X whose initial vertex is v'', whose terminal vertex w is a terminal vertex of Γ_X, and L^i contains e_i. By lemma 5.1.7, the linear subgraph L^i is an O-string, there exists an L^i for $i = 1, 2$ of Γ_X that contains only one edge labelled with 2, and the fundamental group $\pi_1(X)$ is isomorphic to D_{2^k}, where the integer $k > 0$ corresponds to the minimal number of edges with label 2 in all L^3 of Γ_X.

□

We now state our main classification results.

Corollary 5.2.3. *Let Γ be a bicolored pruned trivalent graph. Then $\pi_1(X_\Gamma) \cong \mathbb{Z}_3$ if and only if the following hold:*

1. *The graph Γ is a tree that has exactly one black terminal vertex, all white vertices are genus 0;*
2. *The core reduced graph $\Gamma_C \neq \emptyset$, Γ_C is the core reduced graph of Fig. 4.3, and X_{Γ_C} is a pseudo-projective plane of order 3.*

Proof. Suppose $\pi_1(X_\Gamma) \cong \mathbb{Z}_3$. Since $\pi_1(X_\Gamma)$ is finite the result follows from the proof of theorem 5.2.2. Suppose that condition 1. and 2. holds. Then $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_3$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$.

□

Corollary 5.2.4. *Let Γ be a bicolored pruned trivalent graph. Then $\pi_1(X_\Gamma) \cong \mathbb{Z}_{3*2^k}$ for $k > 0$ if and only if the following hold:*

1. *The graph Γ is a tree that has exactly one black terminal vertex and all white vertices are genus 0;*
2. *The core reduced graph $\Gamma_C \neq \emptyset$ and all edges of Γ_C incident to a terminal white vertex of genus 0 have label 2;*
3. *The graph Γ_C contains exactly one black terminal vertex, all white vertices are genus 0 and have degree ≤ 2, and the graph Γ_C contains no horned trees;*
4. *Let L be an linear subgraph of Γ_C whose initial vertex v is the white vertex adjacent to the black terminal vertex and whose terminal vertex w is a white terminal vertex of Γ_C. Then L is an O-string that contains $r \geq k$ edges with label 2 and there exists at least one L that contains k edges with label 2.*

Proof. Suppose $\pi(X_\Gamma) \cong \mathbb{Z}_{3*2^k}$ for $k > 0$. Since $\pi_1(X_\Gamma)$ is finite the result follows from the proof of theorem 5.2.2.

Suppose that conditions 1. thru 4. holds. By the proof of lemma 5.1.6, $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_{3*2^k}$ for $k > 0$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$.

□

Corollary 5.2.5. *Let Γ be a bicolored pruned trivalent graph. Then $\pi_1(X_\Gamma) \cong \mathbb{Z}_2$ for if and only if either 1.(a)-1.(b) or 2.(a)-2.(e) are satisfied.*

1. (a) *The graph Γ has exactly one white vertex of genus -1 while all other white vertices are genus 0 and all terminal vertices are white;*
   (b) *The core reduced graph $\Gamma_C \neq \emptyset$, Γ_C is a single white vertex of genus -1 with no edges, and X_{Γ_C} is a projective plane;*
2. (a) *The graph Γ contains all white vertices of genus 0 and all terminal vertices are white*
   (b) *The core reduced graph $\Gamma_C \neq \emptyset$ and all edges of Γ_C incident to a terminal vertex of genus 0 have label 2;*
   (c) *The core reduced Γ_C contains all white vertices of genus 0 and all white vertices are of degree ≤ 2, all terminal vertices are white, and Γ_C contains a horned tree H_T.*
   (d) *If L is a linear subgraph of Γ_C whose initial vertex v is a terminal vertex of H_T and whose terminal vertex w is a terminal vertex of Γ_C where $L \cap H_T = v$ and $w \neq v$ then L is an O-string.*
   (e) *The horned tree H_T contains a terminal vertex of Γ_C*

Proof. Suppose $\pi(X_\Gamma) \cong \mathbb{Z}_2$. Since $\pi_1(X_\Gamma)$ is finite the result follows from the proof of theorem 5.2.2.

Suppose that conditions 2.(a)-2.(e) holds. Then by the proof of lemma 5.1.3, $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_2$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$.

Suppose that condition 1.(a)-1.(b) holds. Then $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_2$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$.

□

Corollary 5.2.6. *Let Γ be a bicolored pruned trivalent graph. Then $\pi_1(X_\Gamma) \cong \mathbb{Z}_{2^{k+1}}$ for $k > 0$ if and only if either 1.(a)-(d) or 2.(a)-(d) are satisfied.*

1. (a) *The graph Γ has exactly one white vertex of genus -1 while all other white vertices are genus 0 and all terminal vertices are white*

(b) *The core reduced graph* $\Gamma_C \neq \emptyset$ *and all edges of* Γ_C *incident to a terminal vertex of genus* 0 *have label* 2*;*

(c) *The core subgraph* Γ_C *has exactly one white terminal vertex of genus* -1 *with incident edge label* 1 *while all other white vertices are genus* 0*, all white vertices are of degree* ≤ 2 *and all terminal vertices are white, and* Γ_C *contains no horned trees.*

(d) *Let* L *be a linear subgraph of* Γ_C *whose initial vertex* v *is the white vertex of genus* -1 *and whose terminal vertex* w *is a terminal vertex of* Γ_C *where* $w \neq v$*. Then* L *is an O-string that contains* $r \geq k$ *edges with label 2 and there exists at least one* L *that contains* k *edges with label* 2*.*

2. (a) *The graph* Γ *contains all white vertices of genus* 0 *and all terminal vertices are white*

   (b) *The core reduced graph* $\Gamma_C \neq \emptyset$ *and all edges of* Γ_C *incident to a terminal vertex of genus* 0 *have label* 2*;*

   (c) *The core reduced graph* Γ_C *contains all white vertices of genus* 0 *and are of degree* ≤ 2*, all terminal vertices are white, and* Γ_C *contains a horned tree* H_T*.*

   (d) *Let* L *be a linear subgraph of* Γ_C *whose initial vertex* v *is a terminal vertex of* H_T *and whose terminal vertex* w *is a terminal vertex of* Γ_C *where* $L \cap H_T = v$ *and* $w \neq v$*. Then* L *is an O-string that contains* $r \geq k$ *edges with label 2 and there exists at least one* L *that contains* k *edges with label* 2*.*

Proof. Suppose $\pi(X_\Gamma) \cong \mathbb{Z}_{2^{k+1}}$. Since $\pi_1(X_\Gamma)$ is finite the result follows from the proof of theorem 5.2.2.

Suppose that either conditions 1.(a)-1.(d) or 2.(a)-2.(d) holds. Then by the proof of lemma 5.1.5 or lemma 5.1.3 respectively, $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_{2^{k+1}}$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$.

□

Corollary 5.2.7. *Let* Γ *be a bicolored pruned trivalent graph. Then* $\pi_1(X_\Gamma) \cong D_{2^{k+1}}$ *for* $k \geq 0$ *if and only if the following hold:*

1. *The graph* Γ *is a tree that has all white terminal vertices and white vertices are of genus* 0
2. *The core reduced graph* $\Gamma_C \neq \emptyset$ *and all edges of* Γ_C *incident to a terminal white vertex of genus* 0 *have label* 2*;*
3. *The core reduced graph* Γ_C *has all white vertices of genus* 0 *and all terminal vertices are white, there is exactly one white vertex* v'' *of degree* 3 *while all other white vertices are of degree* ≤ 2*, and* Γ_C *contains no horned tree* H_T *such that either* v'' *and* H_T *are disjoint or* v'' *is a terminal vertex of* H_T

4. *Let L^i be a linear subgraph of Γ_C whose initial vertex is v'', whose terminal vertex w is a terminal vertex of Γ_C, and L^i contains e_i. The linear subgraph L^i is an O-string, there exists an L^i for $i = 1, 2$ of Γ_C that contains only one edge labelled with 2, and all L^3 contains $r \geq k$ edges with label 2 and there exists at least one L^3 that contains k edges with label* 2.

Proof. Suppose $\pi(X_\Gamma) \cong D_{2^{k+1}}$ for $k > 0$. Since $\pi_1(X_\Gamma)$ is finite the result follows from the proof of theorem 5.2.2.

Suppose that either conditions 1-4 holds. Then by the proof of lemma 5.1.7, $\pi_1(X_{\Gamma_C}) \cong D_{2^{k+1}}$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$.

□

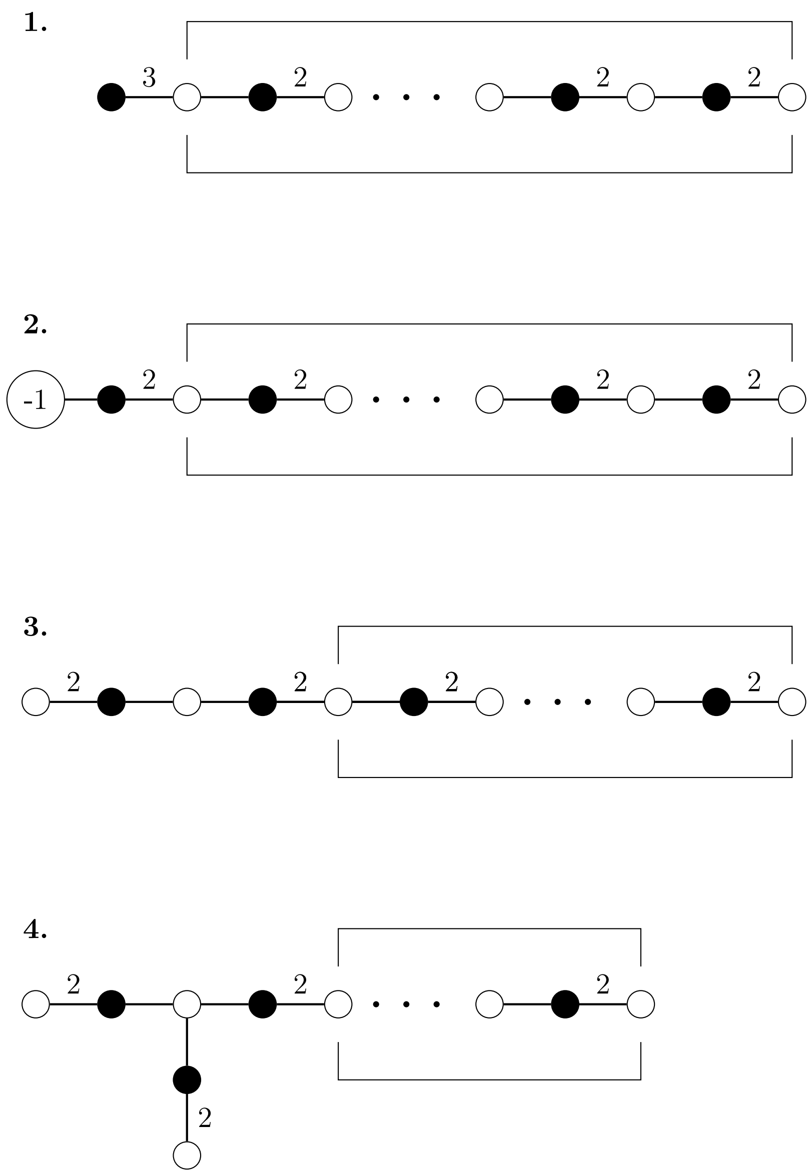


Figure 5.5: Each graph above is core-reduced with no black branch vertices and the boxed in subgraphs are p-strings. Graph 1. satisfies the conditions of lemma 5.2.4. Graph 2. satisfies the conditions of lemma 5.2.6. Graph 3. satisfies the conditions of lemma 5.2.5. Graph 4. satisfies the conditions of lemma 5.2.7.

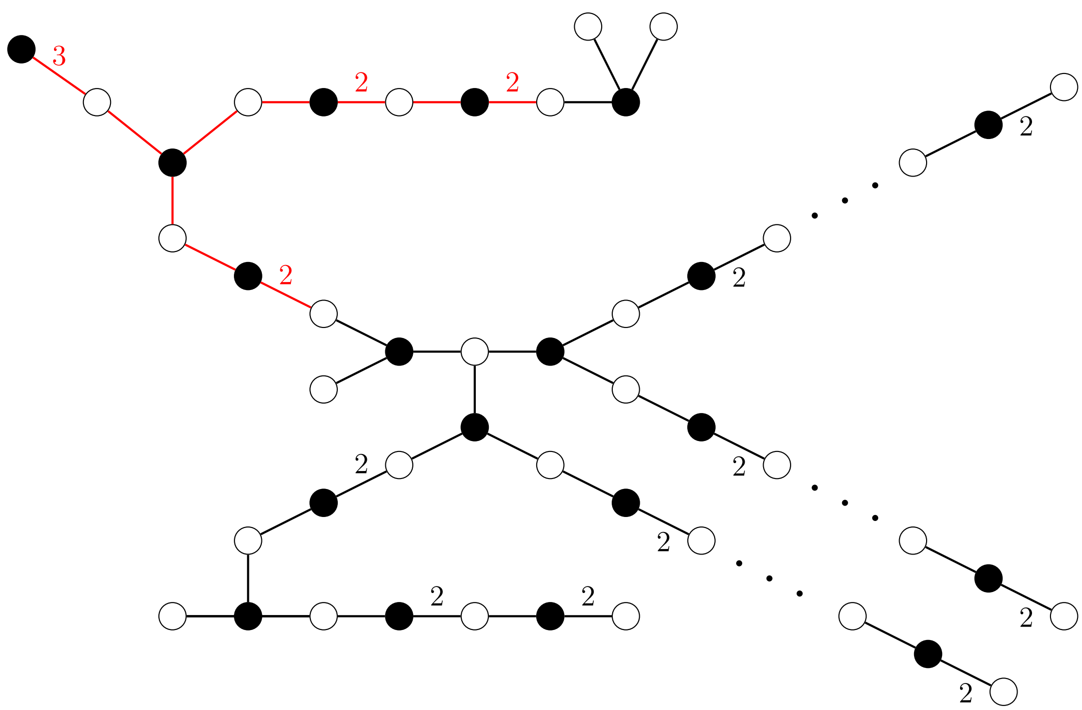


Figure 5.6: A trivalent graph Γ and its core reduced graph Γ_C that satisfies the conditions of corollary 5.2.4. The core reduced graph is composed of the red edges along with incident vertices.

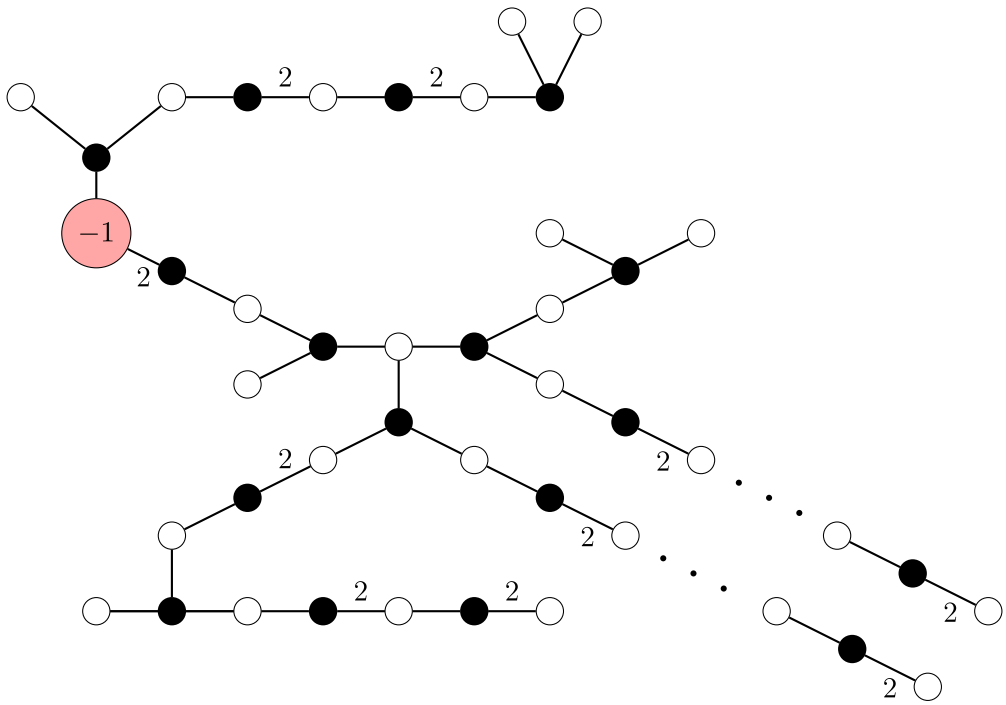


Figure 5.7: A trivalent graph Γ and its core reduced graph Γ_C that satisfies the first set of conditions of corollary 5.2.5. The core reduced graph of Γ is unique and is composed of the red vertex.

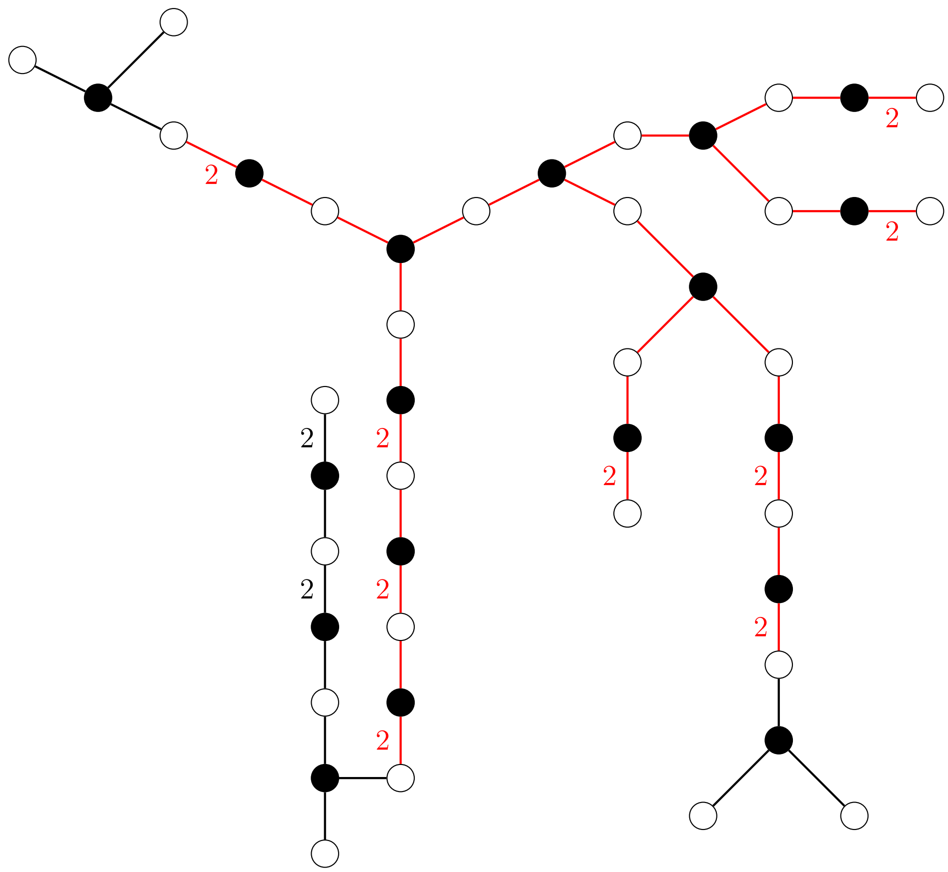


Figure 5.8: A trivalent graph Γ and its core reduced graph Γ_C that satisfies the second set of conditions of corollary 5.2.5. The core reduced graph is composed of the red edges along with incident vertices.

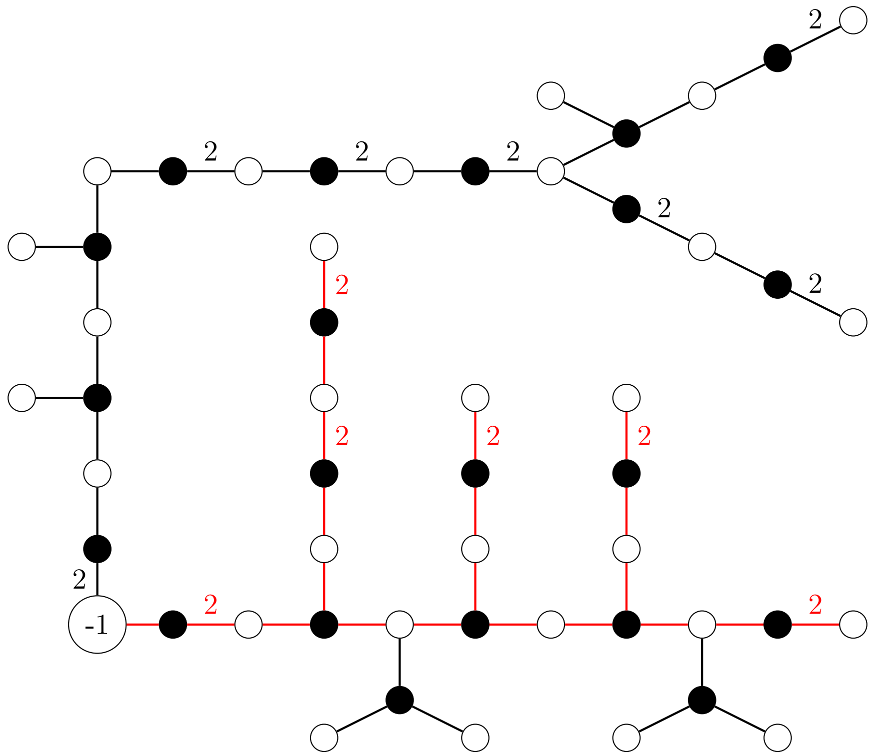


Figure 5.9: A trivalent graph Γ and its core reduced graph Γ_C that satisfies the conditions of corollary 5.2.6. The core reduced graph is composed of the red edges along with incident vertices.

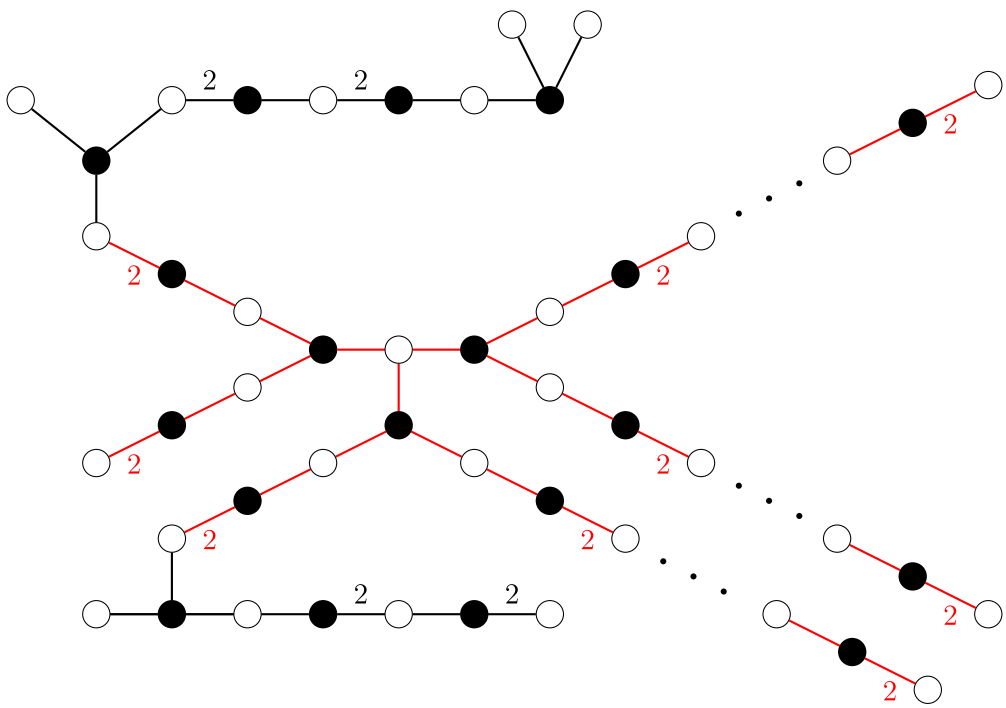


Figure 5.10: A trivalent graph Γ and its core reduced graph Γ_C that satisfies the conditions of corollary 5.2.7. The core reduced graph is composed of the red edges along with incident vertices.

CHAPTER 6

TRIVALENT 2-STRATIFOLDS WITH ABELIAN FUNDAMENTAL GROUP

The finite abelian 2-stratifold groups are the cyclic groups and $\mathbb{Z}_2 \times \mathbb{Z}_2$. A classification of all trivalent labelled graphs that represent trivalent 2-stratifolds with finite abelian fundamental group was given in the previous chapter. The infinite abelian 2-stratifold groups are $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, and $\mathbb{Z} \times \mathbb{Z}_m$. A classification of all trivalent labelled graphs that represent trivalent 2-stratifolds with fundamental group $\mathbb{Z}$ was given in [10]. The main goal of this chapter is to find necessary and sufficient conditions on the graph Γ_X of a trivalent 2-stratifold X so that $\pi_1(X_\Gamma)$ is either $\mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}_m$.

The main work of this chapter will be to obtain a classification of trivalent labelled graphs that represent trivalent 2-stratifolds with $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}_m$ for $m > 1$. This classification is given by theorem 6.3.3. This will lead to a classification of trivalent labelled graphs that represent trivalent 2-stratifolds with $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}$.

6.1 Properties of trivalent 2-stratifolds with abelian fundamental group

First, we review lemma 4 from [10]. Then we state a lemma that follows from the proof of lemma 5 from [10]. These statements will be used to find further necessary conditions on the graph Γ_X of a trivalent 2-stratifold X so that $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}_m$ for $m > 1$.

Lemma 6.1.1. *Let Γ_X be a labelled graph where b is a black vertex of degree $d \geq 2$ such that $r^{-1}(b)$ is contractible in X_Γ. Then $\pi_1(X) = \pi_1(X_{\Gamma_1}) \star \ldots \star \pi_1(X_{\Gamma_n}) \star F_r$ where $\Gamma_1, \ldots, \Gamma_n$ are the components of $\Gamma_X \setminus st(b)$ and F_r is the free group of rank $r = d - n$.*

Lemma 6.1.2. *If X is a 2-stratifold where Γ_X is homeomorphic to S^1 then $\pi_1(X)$ is nonabelian.*

Finally we note the following which also follows from the proof of lemma 5 from [10].

Lemma 6.1.3. *Let X be a 2-stratifold. If $\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}_m$ for $m > 1$ then at least one black vertex belonging to the cycle of Γ_X is a branch vertex.*

Lemmas 6.1.2 and 6.1.3 give the following improvement of lemma 3.4.5.

Corollary 6.1.4. *Let X be a 2-stratifold. If $\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}_m$ for $m > 1$ then Γ_X is homotopy equivalent to S^1 but not homeomorphic to S^1, all white vertices are genus 0 and all terminal vertices are white, and at least one black vertex belonging to the cycle of Γ_X is a branch vertex.*

For Γ_X homotopy equivalent to S^1, we need additional information to determine the homeomorphism class of X. This additional information is an evaluation on Γ_X and was introduced in [10]. We now review this evaluation.

Let κ be a cocycle of $H^1(\Gamma_X, \mathbb{Z}_2) = Hom(H_1(\Gamma_X), \mathbb{Z}_2)$ where $\mathbb{Z}_2 = \{-1, 1\}$. Construct an evaluation λ as follows: take a maximal tree T of Γ_X and let $\lambda(e) = 1$ if e is an edge contained in T and let $\lambda(e) = \kappa([c])$ if e is the edge of $\Gamma_X \setminus T$ and c is the simple cycle of $T \cup e$. With this evaluation, the graph Γ_X along with a cocycle κ uniquely determine X_Γ. In particular, there is at most one (arbitrarily chosen) edge e in the simple closed cycle of Γ_X with $\lambda(e) = -1$.

The graph Γ_X is **nonorientable** if there exists one edge e in the simple closed cycle of Γ_X with $\lambda(e) = -1$. Otherwise, the graph Γ_X is called **orientable** if all edges e of Γ_X have $\lambda(e) = 1$. It will be assumed that a graph Γ_X can either be orientable or nonorientable if not specified.

For Γ_X that is homotopy equivalent to S^1, we refer to the subgraph that is homeomorphic to S^1 as the cycle C of Γ_X.

6.2 Graphs of trivalent 2-stratifolds with abelian fundamental group

In this section, we find further necessary conditions on Γ_X so that $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}_m$ where $m > 1$. Then for X_Γ whose associated bipartite labelled graphs Γ_X satisfy these necessary conditions, we show that if $\pi_1(X_\Gamma)$ is abelian then $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}_{2^k}$ where $k \geq 1$. This is done in theorem 6.2.7.

In this section, it is assumed, unless otherwise noted, that all 2-stratifolds X have an associated graph Γ_X **that is homotopy equivalent to S^1 but not homeomorphic to S^1, all white vertices are genus 0 and all terminal vertices are white, and at least one black vertex belonging to the cycle C of Γ_X is a branch vertex.** By corollary 6.1.4, these are necessary conditions on X for X to have fundamental group $\mathbb{Z} \times \mathbb{Z}_m$ where $m > 1$.

Corollary 6.2.1. *Let X be a pruned trivalent 2-stratifold. If the fundamental group of X is $\mathbb{Z} \times \mathbb{Z}_n$ for $n > 1$ then the cycle C of Γ_X contains no black branch vertex b where $r^{-1}(b)$ is contractible in X_Γ.*

Proof. Suppose that the cycle C of Γ_X contains a black branch vertex b where $r^{-1}(b)$ is contractible in X_Γ. Then $\Gamma_X \setminus st(b)$ contains two components Γ_1 and Γ_2. By lemma 6.1.1, if at least one X_{Γ_i} has nontrivial fundamental group then $\pi_1(X)$ is nonabelian and if both X_{Γ_1} and X_{Γ_2} are simply connected then $\pi_1(X) \cong \mathbb{Z}$. □

Lemma 6.2.2. *Let X be a pruned trivalent 2-stratifold where the graph Γ_X has a label 2 for all edges incident to a terminal vertex. Then X has nonabelian fundamental group if Γ_X contains at least one of the following:*

1. *a horned tree;*
2. *a white vertex w of degree > 2 contained in $\Gamma_X \setminus C$.*

Proof. (1.) Suppose that Γ_X contains a horned tree H. Let T be a maximal tree of Γ that contains H. Let the white vertex and the black vertex incident to e be called w and b respectively where e is the edge of Γ_X that is not contained in T. Then b is disjoint from H. Attach to the black vertex b a white vertex of genus 0 with edge label 1. The black vertex b in resulting graph Γ' corresponds to the contractible curve $r^{-1}(b)$ in $X_{\Gamma'}$. Let the components of $\Gamma' \setminus st(b)$ be called Γ'_1 and Γ'_2 where H is contained in Γ'_1. Then $X_{\Gamma'_1}$ has nontrivial fundamental group. By lemma 6.1.1, the fundamental group of Γ' is isomorphic to $\pi_1(X_{\Gamma'_1}) \star \pi_1(X_{\Gamma'_2}) \star \mathbb{Z}$. Therefore $\pi_1(X_\Gamma)$ surjects onto a nonabelian group.

(2.) Suppose that w is a white vertex of degree 3 contained in $\Gamma_X \setminus C$. Let L be the linear subgraph of Γ_X with terminal vertices w and v where v is contained in C such that $L \cap C = v$. Let e_1 be the edge incident to w that is contained in L. Allow P to be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus \{e_1\}$ that contains w. If Γ_X is pruned at P, the resulting graph P' is a tree that contains all white terminal vertices with incident edge label 2. Then $X_{P'}$ has nontrivial fundamental group by lemma 4.3.2. Attach white vertices of genus 0 with edge label 1 to all black vertices contained not contained in P. Then $\pi_1(X)$ surjects onto $\pi_1(X_{P'}) \cong \pi_1(X_{P'}) \star \mathbb{Z}$.

□

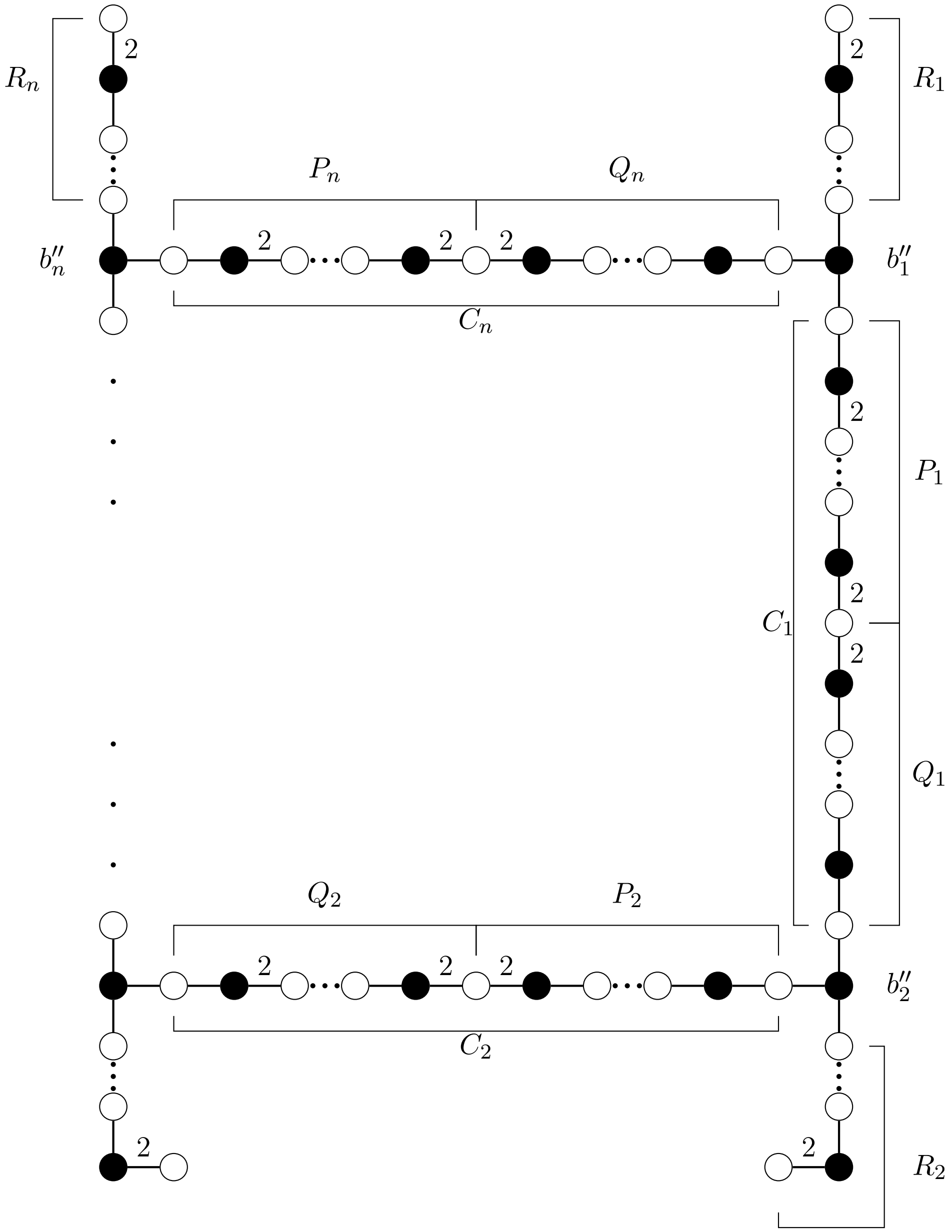


Figure 6.1: All R_i are p-strings.The graph Γ is an echinus graph.

We introduce some notation that will be used throughout the section.

Notation 6.2.1. *Consider X to be a pruned trivalent 2-stratifold where Γ_X has a label 2 for all edges incident to a terminal vertex.*

For a fixed orientation of C, the successive black branch vertices will be denoted $b''_1, \ldots, b''_n$ where $n \geq 1$. The adjacent white vertex to b''_i not contained in C will be denoted v''_i. The subgraphs of Γ_X corresponding to the components of $\Gamma_X \setminus (C \cup \bigcup st(b''_i))$ will be denoted $R_1, \ldots, R_n$ where R_i contains the white vertex v''_i. The successive subgraphs corresponding to the components of $\Gamma_X \setminus (\bigcup R_i \cup \bigcup st(b''_i))$ will be denoted C_i.

If L is a 1-connected trivalent 2-stratifold where Γ_L is a linear graph then Γ_L contains no horned trees. Then Γ_L is either a p-string, a q-string, or a p-string followed by a q-string. We define $L[p, q]$ to be a linear graph consisting of a p-string of length $2p \geq 0$ followed by a q-string of length $2q \geq 0$. If both the p-string and q-string contained in $L[p, q]$ have length 0 then $L[p, q]$ is a white vertex of genus 0.

Let Γ_X satisfy the conditions of corollary 6.1.4. Further let Γ_X have a label 2 for all edges incident to a terminal vertex and contain only white vertices of degree ≤ 2. By lemma 6.2.2, if $\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}_n$ then Γ_X contains no horned trees. Each C_i is a linear subgraph that has white terminal vertices and contains no horned trees. Then C_i is a $L[p, q]$ graph.

Lemma 6.2.3. *Let X be a pruned trivalent 2-stratifold where Γ_X has a label 2 for all edges incident to a terminal vertex. If $\pi_1(X)$ is abelian then the following hold:*

1. *Let L be a linear subgraph of Γ_X contained in R_i whose initial vertex is v''_i and whose terminal vertex is a terminal vertex of Γ_X where $L \cap C = \emptyset$. Then L is an O-string.*

2. *Let $k > 0$ be the minimum number of edges with label 2 in all linear subgraphs L contained in R_i whose initial vertex is v''_i and whose terminal vertex is a terminal vertex of Γ_X where $L \cap C = \emptyset$. Let Γ' be obtained from Γ_X by replacing R_i with a p-string of length $2k$. Then $\pi_1(X_\Gamma) \cong (X_{\Gamma'})$.*

Proof. By lemma 6.2.2, all white vertices in R_i are of degree ≤ 2.

(1.) Let L be a linear subgraph of R_i whose initial vertex is v''_i and whose terminal vertex v is a terminal vertex of Γ_X where $L \cap C = \emptyset$. Suppose L is not an O-string. Order the vertices of L as $w_1 - b_2 - ... - b_r - w_r$ so that the initial vertex $w_1 = v''_i$ and $w_r = v$.

Then either the subgraph $w_{r-2} - b_{r-1} - w_{r-1}$ has successive edges with label $m_{r-1} = 2$ and $n_{r-1} = 1$ or there exists a subgraph $w_{j-1} - b_j - w_j$ that has successive edges with label $m_j = 2$ and $n_j = 1$ where $1 < j < r-1$ and for all $j < k < r$ the successive labels m_k, n_k are either $m_k = 1$ and $n_k = 2$ or $m_k = 1$ and $n_k = 1$. If the subgraph $w_{r-2} - b_{r-1} - w_{r-1}$ of L has successive edge labels $m_{r-1} = 2$ and $n_{r-1} = 1$ then Γ_X contains a horned tree.

Suppose the subgraph $w_{j-1} - b_j - w_j$ has successive edges with label $m_j = 2$ and $n_j = 1$ where $1 < j < r-1$ and for all $j < k < r$ the successive labels m_k, n_k are either $m_k = 1$ and $n_k = 2$ or $m_k = 1$ and $n_k = 1$. Let e be the edge incident to w_{j-1} that is not incident to b_j. Let K be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus \{e\}$ that contains w_{j-1}. If Γ_X is pruned at K, the resulting graph K' is a tree that contains all white terminal vertices with incident edge label 2. By lemma 4.3.2, $\pi_1(X_{K'})$ is nontrivial. For the graph Γ_X, attach a white vertex of genus 0 with an edge of label 1 for all black vertices not contained in K. Then there is an epimorphism from $\pi_1(X) \to \pi_1(X_{K'}) \star \mathbb{Z}$.

(2.) Suppose that R_i contains no black vertices of degree 3. Then R_i is a p-string otherwise $\pi_1(X_\Gamma)$ is nonabelian.

Suppose that R_i contains 1 black vertex of degree 3. Let b be the black vertex of degree 3 contained in R_i that is adjacent to the vertices v_1, v_2, v_3 such that v_j is the initial vertex of a terminal linear subgraph T_j for $j = 1, 2$. The linear subgraphs T_1, T_2 are p-strings. Let the terminal vertex of T_j be called t_j. The linear subgraph L_j of Γ_X with initial vertex v_i'' and terminal vertex t_j is an O-string with $k_j > 0$ edges with label 2. Apply operation $B1$ to $st(b) \cup T_1 \cup T_2$ and let the resulting graph be Γ'. In the graph Γ', let the associated p-string be called T' and let the terminal vertex of T' and Γ' be called t'. The linear subgraph L' of Γ' with initial vertex is v_i'' and terminal vertex t' is a p-string of length $2k > 0$ where $k = min\{k_1, k_2\}$. The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$.

Suppose that R_i contains $n > 1$ black vertices of degree 3. Let b be the black vertex of degree 3 contained in R_i that is adjacent to the vertices v_1, v_2, v_3 such that v_j is the initial vertex of a terminal linear subgraph T_j for $j = 1, 2$. Then the linear subgraph T_1, T_2 are p-strings. Let the terminal vertex of T_j be called t_j. Apply operation $B1$ to $st(b) \cup T_1 \cup T_2$ and let the resulting graph be Γ'. Let R_i' of Γ' be the subgraph that corresponds to R_i in Γ. The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$.

The linear subgraph L_j of Γ_X with initial vertex v_i'' and terminal vertex t_j is an O-string with $k_j > 0$ edges with label 2. In the graph Γ', let the associated p-string be called T' and let the terminal vertex of T' and Γ' be called t'. The linear subgraph L' of Γ' with initial vertex is v_i'' and terminal vertex t' is an O-string that contains $k > 0$ edges with label 2 where $k = min\{k_1, k_2\}$.

There exists an O-string O' contained in R_i whose initial vertex is v_i'' and whose terminal vertex is a terminal vertex of Γ_X where the number of edges k' with label 2 is minimal. If O' is disjoint from $st(b) \cup T_1 \cup T_2$ then O' is contained in R_i' of Γ'. If O' is not disjoint from $st(b) \cup T_1 \cup T_2$ then L' has $k = k'$ edges with label 2.

□

An **echinus graph** $E = E[p_1, q_1, r_1; \ldots; p_n, q_n, r_n]$ is a trivalent labelled graph Γ with the following properties:

1. Γ is homotopy equivalent to S^1 but not homeomorphic to S^1.
2. All vertices of Γ are of degree 2, except for $n \geq 1$ black branch vertices of the cycle C of Γ.
3. Each C_i is the linear graph $L[p_i, q_i]$ with $p_i, q_i \geq 0$ for $i = 1, \ldots, n$.
4. Each R_i is a p-string of length $2r_i > 0$ for $i = 1, \ldots, n$.

An example of an echinus graph is seen in figure 6.1. For an echinus graph $E[p_1, q_1, r_1; \ldots; p_n, q_n, r_n]$ the fundamental group of X_E has the following presentation:

$$\{b_1, \ldots, b_n, t : b_j^{2^{r_j}} = 1, b_i^{2p_i} = b_{i+1}^{2q_i}, tb_n^{2p_n}t^{-1} = b_1^{\epsilon 2q_n}, j = 1, \ldots, n, i = 1, \ldots, n-1\}$$

where b_i are the generators corresponding to the black branch vertex b_i'' of E for a fixed orientation of C, $\epsilon = 1$ if E is orientable and $\epsilon = -1$ if E is nonorientable.

We show that attaching a p-string to echinus graph results in a 2-stratifold with nonabelian fundamental group. This will be used to show that if $\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}_n$ then Γ_X contains no white vertices of degree > 2.

Lemma 6.2.4. *Let E be an echinus graph where $E = E[p_1, q_1, r_1; \ldots; p_n, q_n, r_n]$ and all $r_i = 1$. Let $\bar{E}$ be obtained from E by attaching the linear graph $-b' - w'$ with successive edge labels $1, 2$ to a white vertex of C. Then $X_{\bar{E}}$ has nonabelian fundamental group.*

Proof. For the echinus graph there are three main cases to consider: $p_1 + \ldots + p_n = 0$ and $q_1 + \ldots + q_n = 0$; $p_1 + \ldots + p_n \neq 0$ and $q_1 + \ldots + q_n \neq 0$; $p_1 + \ldots + p_n = 0$ and $q_1 + \ldots + q_n \neq 0$ or $p_1 + \ldots + p_n \neq 0$ and $q_1 + \ldots + q_n = 0$.

Case 1. Suppose that $p_1 + \ldots + p_n = 0$ and $q_1 + \ldots + q_n = 0$ in E. Fix an orientation on C of $\bar{E}$ such that the white vertex of degree 3 is adjacent to b_1'' and b_n''. The fundamental group of $X_{\bar{E}}$ has the following presentation:

$$\{b_1, \ldots, b_n, t, c_1, c_2, c_3, : c_1 = b_1, b_i = b_{i+1}, tb_nt^{-1} = c_2^{\epsilon}, b_j^2, c_3^2, c_1c_2c_3, j = 1, \ldots, n, i = 1, \ldots n-1\}$$

where $\epsilon = \pm 1$. Note that $c_2 = c_2^{-1}$. Then this presentation is equivalent to the following:

$$\{t, c_1, c_2, c_3, : tc_1t^{-1} = c_2, c_1^2, c_2^2, c_3^2, c_1c_2c_3\}.$$

The fundamental group of $X_{\bar{E}}$ is then an HNN extension of a dihedral group along proper subgroups. By corollary 3.1.3, the group $\pi_1(X_{\bar{E}})$ is nonabelian.

Case 2. Suppose that $p_1 + \ldots + p_n \neq 0$ and $q_1 + \ldots + q_n = 0$ in E. (If $p_1 + \ldots + p_i = 0$ and $q_1 + \ldots + q_n \neq 0$ then the same proof applies.) Let w be the white vertex of degree 3. Fix an orientation on C of $\bar{E}$ such that w is contained in C_1.

Suppose that $p_1 = 0$. The fundamental group of $X_{\bar{E}}$ has the following presentation where the generators are given by $\mathcal{G} = \{b_1, \ldots, b_n, t, c_1, c_2, c_3\}$ and the relations, $\mathcal{R}$, are given by the following:

$$\mathcal{R} = \{b_1 = c_1, c_2 = b_2, b_i^{2^{p_i}} = b_{i+1}, tb_n^{2^{p_n}}t^{-1} = b_1^{\epsilon}, {b_j}^2, c_3^2, c_1c_2c_3, i = 2, \ldots, n-1, j = 1, \ldots, n\}.$$

There exists a $p_i > 0$ for $i > 1$ where either $b_i^{2^{p_i}} = b_{i+1}$ or $tb_n^{2^{p_n}}t^{-1} = b_1^{\epsilon}$. Since each b_j has order 2 then $b_{i+1} = 1$ or $b_1 = 1$ respectively. Then at least one of the following curves $r^{-1}(b_{i+1}'')$, $r^{-1}(b_1'')$ is contractible. Let the black vertex contained in C corresponding to the contractible curve be called b. Let the components of $\bar{E} \setminus st(b)$ be called Γ_1 and Γ_2. By lemma 6.1.1, the fundamental group of $X_{\bar{E}}$ is isomorphic to $\pi_1(X_{\Gamma_1}) \star \pi_1(X_{\Gamma_2}) \star \mathbb{Z}$. If $\pi_1(X_{\Gamma_1})$ and $\pi_1(X_{\Gamma_2})$ are trivial then $X_{\bar{E}}$ has infinite cyclic fundamental group. This contradicts Lemma 6 of [10] (If $\pi_1(X) \cong \mathbb{Z}$ where Γ_X contains all terminal edges with label 2 then Γ_X contains no white vertices of degree > 2). Therefore at least one of $\pi_1(X_{\Gamma_1'})$, $\pi_1(X_{\Gamma_2'})$ is nontrivial. Then $X_{\bar{E}}$ has nonabelian fundamental group.

Suppose that $p_1 > 0$. Let w_0 be the white vertex adjacent to b''_1 contained in C_1 and let w'_0 be the white vertex adjacent to b''_2 contained in C_1. Then the linear subgraph with initial vertex w_0 and terminal vertex w is a p-string of length $2p'_1$ and the linear subgraph with initial vertex w and terminal vertex w'_0 is a p-string of length $2p''_1$. The fundamental group of $X_{\bar{E}}$ has the following presentation where the generators are $\mathcal{G} = \{b_1, \ldots, b_n, t, c_1, c_2, c_3\}$ and the relations, $\mathcal{R}$, are given by the following:

$$\mathcal{R} = \{b_1^{2^{p'_1}} = c_1, c_2^{2^{p''_1}} = b_2, b_i^{2^{p_i}} = b_{i+1}, tb_n^{2^{p_n}}t^{-1} = b_1^{\epsilon}, {b_j}^2, c_3^2, c_1c_2c_3, i = 2, \ldots, n-1, j = 1, \ldots, n\}$$

If there exists a $p_i > 0$ for $i > 1$ where either $b_i^{2^{p_i}} = b_{i+1}$ or $tb_n^{2^{p_n}}t^{-1} = b_1^{\epsilon}$ then $b_{i+1} = 1$ or $b_1 = 1$. Then at least one of the following curves $r^{-1}(b''_{i+1})$, $r^{-1}(b''_1)$ are contractible. By the previous case, $X_{\bar{E}}$ has nonabelian fundamental group.

We assume all $p_i = 0$ if $i \neq 1$. Then the subgraph E of $\bar{E}$ has the given labellings $E = E[p_1, 0, 1; 0, 0, 1; \ldots; 0, 0, 1; 0, 0, 1]$.

Case 2.a. Suppose $w = w_0$ and $p_1 > 0$. The fundamental group of $X_{\bar{E}}$ has the following presentation where the generators are given by $\mathcal{G} = \{b_1, \ldots, b_n, t, c_1, c_2, c_3\}$ and the relations are given by:

$$\mathcal{R} = \{b_1 = c_1, {c_2}^{2^{p_1}} = b_2, b_i = b_{i+1}, tb_nt^{-1} = b_1^{\epsilon}, {b_j}^2, c_3^2, c_1c_2c_3, j = 1, \ldots, n, i = 2, \ldots, n-1\}.$$

This presentation is equivalent to the following:

$$\{t, c_1, c_2, c_3, : tc_2^{2^{p_1}}t^{-1} = c_1, c_1^2, c_2^{2^{p_1+1}}, c_3^2, c_1c_2c_3\}.$$

Then $\pi_1(X_{\bar{E}})$ surjects onto the following nontrivial free product:

$$\{t, c_2, c_3, : c_2^2, c_3^2, c_2c_3\}.$$

Case 2.b Suppose $w = w'_0$ and $p_1 > 0$. The fundamental group of $X_{\bar{E}}$ has the following presentation where the generators are given by $\mathcal{G} = \{b_1, \ldots, b_n, t, c_1, c_2, c_3\}$ and the relations are given by:

$$\mathcal{R} = \{b_1^{2p_1} = c_1, c_2 = b_2, b_i = b_{i+1}, tb_nt^{-1} = b_1^{\epsilon}, {b_j}^2, c_3^2, c_1c_2c_3, j = 1, \ldots, n, i = 2, \ldots, n-1\}.$$

This presentation is equivalent to the following:

$$\{t, c_2 : {c_2}^2\}.$$

Case 2.c. Suppose w is not adjacent to the black vertex b_1'' and is not adjacent to the black vertex b_2'' and $p_1 > 1$. The fundamental group of $X_{\bar{E}}$ has the following presentation where the generators are given by $\mathcal{G} = \{b_1, \ldots, b_n, t, c_1, c_2, c_3\}$ and the relations are given by:

$$\mathcal{R} = \{b_1^{2^{p_1'}} = c_1, c_2^{2^{p_1''}} = b_2, b_i = b_{i+1}, tb_nt^{-1} = b_1^{\epsilon}, {b_j}^2, c_3^2, c_1c_2c_3, j = 1, \ldots, n, i = 2, \ldots, n-1\},$$

where $p_1' > 0$ and $p_1'' > 0$. This presentation is equivalent to the following:

$$\{t, c_2 : c_2^2\}.$$

Case 3. Suppose that $p_1 + \ldots + p_n \neq 0$ and $q_1 + \ldots + q_n \neq 0$. We will show that the echinus graph E has a 2-stratifold X_E that has a nonabelian fundamental group. Then it follows that $X_{\bar{E}}$ has a nonabelian fundamental group.

Let E be the following echinus graph $E = E[p_1, q_1, r_1; \ldots; p_n, q_n, r_n]$. By proposition 5 of [10], $\pi_1(X_E)$ is not isomorphic to $\mathbb{Z}$. By assumption all $r_i = 1$, if $p_i > 1$ (or $q_i > 1$) then replacing p_i with 1 (resp. q_i with 1) does not alter the group $\pi_1(X_E)$. If $p_i = q_i = 0$ for some i in $1 \leq i \leq n-1$ then the echinus graph $E' = E'[p_1, q_1, r_1; \ldots; p_{i-1}, q_{i-1}, r_{i-1}; p_{i+1}, q_{i+1}, r_{i+1}; \ldots; p_n, q_n, r_n]$ has a 2-stratifold $X_{E'}$ where $\pi_1(X_{E'}) \cong \pi_1(X_E)$.

For the echinus graph E, we assume that there does not exist an i for $1 \leq i \leq n-1$ such that $p_i = q_i = 0$. We also assume that all p_i, q_i are either 0 or 1.

The fundamental group of X_E has the following presentation:

$$\{b_1, \ldots, b_n, t : b_j^2 = 1, b_i^{2p_i} = b_{i+1}^{2q_i}, tb_n^{2p_n}t^{-1} = b_1^{\epsilon 2q_n}, j = 1, \ldots, n, i = 1, \ldots, n-1\}.$$

If there exists a $p_i = 1$ and $q_i = 0$ then the generator $b_{i+1} = 1$ if $i = 1, \ldots, n-1$ or the generator $b_1 = 1$ if $i = n$. Then either the curve $r^{-1}(b_{i+1}'')$ is contractible or the curve $r^{-1}(b_1'')$ is contractible.

Let the black vertex contained in C corresponding to the contractible curve be called b. Let the components of $\Gamma' \setminus st(b)$ be called Γ'_1 and Γ'_2. By lemma 6.1.1, the fundamental group of X_E is isomorphic to $\pi_1(X_{\Gamma'_1}) \star \pi_1(X_{\Gamma'_2}) \star \mathbb{Z}$. At least one of $\pi_1(X_{\Gamma'_1})$, $\pi_1(X_{\Gamma'_2})$ is nontrivial. Then X_E has nonabelian fundamental group. Similarly, if there exists a $p_i = 0$ and $q_i = 1$ then X_E has nonabelian fundamental group.

We assume that $p_i = 1$ and $q_i = 1$ for $1 \leq i \leq n-1$. Then $E = E[1,1,1;\ldots;1,1,1]$ or $E = E[1,1,1;\ldots;0,0,1]$. If $n > 2$ then E contains a horned tree. If $n = 2$ and $E = E[1,1,1;1,1,1]$ then E contains a horned tree. The last two cases are when $E = E[1,1,1]$ or $E = E[1,1,1;0,0,1]$

If $E = E[1,1,1]$ then the fundamental group of X_E has the following presentation:

$$\{b_1, t : b_1^2, tb_1^2t^{-1} = b_1^2\}.$$

If $E = E[1,1,1;0,0,1]$ then the fundamental group of X_E has the following presentation:

$$\{b_1, b_2, t : b_1^2, b_2^2, tb_2t^{-1} = b_1\}.$$

This is equivalent to

$$\{b_2, t : b_2^2\}.$$

□

The next statement follows from case 3 of the previous lemma.

Corollary 6.2.5. *Let E be an echinus graph where $E = E[p_1, q_1, r_1; \ldots; p_n, q_n, r_n]$. If $p_1 + \ldots + p_n \neq 0$ and $q_1 + \ldots + q_n \neq 0$ then $\pi_1(X_E)$ is nonabelian.*

Lemma 6.2.6. *Let X be a pruned trivalent 2-stratifold where the graph Γ_X has a label 2 for all edges incident to a terminal vertex. If $\pi_1(X) = \mathbb{Z} \times \mathbb{Z}_m$ for $m > 1$ then the following holds:*

1. *All white vertices of Γ_X contained in the cycle C are of degree ≤ 2.*

2. *If b is a black vertex contained in the cycle C then it is a branch vertex.*

Proof. By lemma 6.2.2 all white vertices in R_i are of degree ≤ 2.

(1.) Suppose that Γ_X contains a white vertex w of degree 3 such that w is contained in C and all other white vertices in Γ_X are of degree ≤ 2.

For each R_i of Γ_X, let $k_i > 0$ be the minimum number of edges with label 2 in all linear subgraphs L contained in R_i whose initial vertex is v_i'' and whose terminal vertex is a terminal vertex of Γ_X where $L \cap C = \emptyset$. If R_i is not a p-string of length $2k_i$ then replace R_i with a p-string of length $2k_i$. Let the resulting graph be called Γ'. Then $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$.

Let e be the edge incident to w that is not contained in C of Γ'. Let K be the subgraph of Γ' that corresponds to the component of $\Gamma' \setminus e$ that contains the cycle C and let G be the subgraph of Γ' that corresponds to the component of $\Gamma' \setminus e$ that is disjoint from the cycle C. Prune Γ' at K. The resulting graph K' is an echinus graph. Then each C_i restricted to the subgraph K of Γ' is a linear $L[p_i, q_i]$ graph.

Suppose the subgraph $P = w \cup e \cup G$ of Γ' has no black vertices of degree 3. Then P is a terminal p-string of Γ' where $P \cap C = w$. Let T_i be the p-string of length 2 contained in Γ' with initial vertex v_i'' where $T_i \cap C = \emptyset$. Let L be the p-string of length 2 contained in Γ' with initial vertex w where $L \cap C = w$. Prune Γ' at $\bigcup T_i \cup \bigcup st(b_i) \cup C \cup L$ and let the resulting graph be $\bar{E}$. By lemma 6.2.4, $X_{\bar{E}}$ has nonabelian fundamental group.

Suppose that P has $k > 0$ black vertices of degree 3. Let b be a black vertex of degree 3 contained in P where b is adjacent to the vertices v_1, v_2, v_3 such that v_j is the initial vertex of a terminal linear subgraph T_j for $i = 1, 2$. If T_j contains a horned tree then $X_{\Gamma'}$ has nonabelian fundamental group. We assume that the terminal linear subgraphs T_j are p-strings. Apply operation $B1$ on $st(b) \cup T_1 \cup T_2$ such that the resulting graph Γ'' contains a subgraph P' with $k-1$ black vertices of degree 3 and $\pi_1(X_{\Gamma'}) \cong \pi_1(X_{\Gamma''})$. By induction hypothesis, the result holds.

(2.) Assume all white vertices contained in C are of degree ≤ 2. If X has abelian fundamental group then each C_i is a $L[p_i, q_i]$ graph. For each R_i, let $k_i > 0$ be the minimum number of edges with label 2 in all linear subgraphs L contained in R_i whose initial vertex is v_i'' and whose terminal vertex is a terminal vertex of Γ_X where $L \cap C = \emptyset$. If R_i is not a p-string of length $2k_i$ then replace R_i with a p-string of length $2k_i$. Let the resulting graph be called Γ'. Then Γ' is an echinus graph where $\Gamma' = E[p_1, q_1, r_1; \ldots; p_n, q_n, r_n]$ and $\pi_1(X_{\Gamma'}) \cong \pi_1(X_\Gamma)$.

Suppose that there is a black vertex contained in C of Γ' that is not a branch vertex. Then exactly one of the following cases occur: $p_1 + \ldots + p_n = 0$ and $q_1 + \ldots + q_n \neq 0$; $p_1 + \ldots + p_n \neq 0$ and $q_1 + \ldots + q_n = 0$; or $p_1 + \ldots + p_n \neq 0$ and $q_1 + \ldots + q_n \neq 0$. If either $p_1 + \ldots + p_n = 0$ and

$q_1 + \ldots + q_n \neq 0$ or $p_1 + \ldots + p_n \neq 0$ and $q_1 + \ldots + q_n = 0$ then $\pi_1(X_{\Gamma'}) \cong \mathbb{Z}$ by proposition 5 of [10]. If $p_1 + \ldots + p_n \neq 0$ and $q_1 + \ldots + q_n \neq 0$ then $\pi_1(X_{\Gamma'})$ is nonabelian by corollary 6.2.5.

□

Theorem 6.2.7. *Let X be a pruned trivalent 2-stratifold where the graph Γ_X has a label 2 for all edges incident to a terminal vertex. If $\pi_1(X)$ is $\mathbb{Z} \times \mathbb{Z}_m$ for $m > 1$ then all of the following hold:*

1. *If b is a black vertex contained in the cycle C then it is a branch vertex;*
2. *Γ_X contains no horned trees and all white vertices are of degree ≤ 2;*
3. *If L is a linear subgraph of Γ_X whose initial vertex is v_i'' and whose terminal vertex is a terminal vertex of Γ_X where $L \cap C = \emptyset$ then L is an O-string;*
4. *The fundamental group $\pi_1(X)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_{2^k}$ for $k \geq 1$ where Γ_X is orientable if $k > 1$ otherwise Γ_X is either orientable or nonorientable. The integer $k \geq 1$ corresponds to the minimal number of edges with label 2 in all linear subgraphs L whose initial vertex is v_i'' and whose terminal vertex is a terminal vertex of Γ_X where $L \cap C = \emptyset$ for $1 \leq i \leq n$.*

Proof. Suppose $\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}_m$ for $m > 1$. Then (1.) follows from lemma 6.2.6, (2.) follows from lemma 6.2.2 and lemma 6.2.6, and (3.) follows from lemma 6.2.3.

For each R_i, let $r_i > 0$ be the minimum number of edges with label 2 in all linear subgraphs L contained in R_i whose initial vertex is v_i'' and whose terminal vertex is a terminal vertex of Γ_X where $L \cap C = \emptyset$. If R_i is not linear then then replace R_i with a p-string of length $2r_i$. Let the resulting graph be called Γ'. Then Γ' is an echinus graph where $\Gamma' = E[p_1, q_1, r_1; \ldots; p_n, q_n, r_n]$ where $p_i = q_i = 0$ for all i. Then $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$ and the fundamental group of $X_{\Gamma'}$ has the following presentation:

$$\{b_1, \ldots, b_n, t : b_j^{2^{k_j}} = 1, b_i = b_{i+1}, tb_nt^{-1} = b_1^{\epsilon}, j = 1, \ldots, n, i = 1, \ldots, n-1\}.$$

where ϵ is -1 if Γ is nonorientable otherwise ϵ is 1. Let k be the minimum of $\{k_1, \ldots, k_n\}$. Then the fundamental group of X_Γ admits the following presentation:

$$\{b_1, t : b_1^{2^k} = 1, tb_1t^{-1} = b_1^{\epsilon}\}.$$

If $k = 1$ then $\pi_1(X_\Gamma)$ is $\mathbb{Z} \times \mathbb{Z}_2$ and Γ_X is either orientable or nonorientable.

Suppose $k > 1$. Then $\epsilon = 1$ if $\pi_1(X_\Gamma)$ is abelian. Then $\pi_1(X_\Gamma)$ is $\mathbb{Z} \times \mathbb{Z}_{2^k}$ and the graph Γ_X is orientable.

□

6.3 Labellings of trivalent 2-stratifolds with abelian fundamental group

For a trivalent bicolored graph Γ, we now describe the necessary and sufficient conditions on Γ for $\pi_1(X_\Gamma)$ to be isomorphic to $\mathbb{Z} \times \mathbb{Z}_m$ where $\Gamma = \Gamma_X$.

It is assumed throughout this section, unless otherwise noted, that all 2-stratifolds X have an associated graph Γ_X **that is homotopy equivalent to S^1 but not homeomorphic to S^1, all white vertices are genus 0 and all terminal vertices are white, and at least one black vertex belonging to the cycle C of Γ_X is a branch vertex. It is further assumed that Γ_X is pruned.**

First, we review the definition of a core-reduced graph that was introduced in [10].

If the graph $\Gamma \setminus C$ does not contain a black branch vertex of distance 1 to a terminal vertex then Γ is core-reduced. If $\Gamma \setminus C$ contains a black branch vertex of distance 1 to a terminal vertex we let $B = \{b_{01}, \ldots, b_{0k}\}$ be the set of all outermost black branch vertices where each b_{0i} has distance 1 from a terminal vertex w_{0i}. Let the component of $\Gamma \setminus (st(b_{0i}) \cup w_{0i})$ that is disjoint from C be denoted T_{0i}. If there exists at least one component T_{0i} that is not 1-connected let $\Gamma_0 = \emptyset$. If each T_{0i} is 1-connected then let $\Gamma'_0 = \Gamma \setminus (\bigcup st(b_{0i}) \cup \bigcup w_{0i} \cup \bigcup T_{0i})$. If Γ'_0 is pruned then let $\Gamma_0 = \Gamma'_0$, otherwise let Γ_0 be the pruned Γ'_0. For $\Gamma_0 \neq \emptyset$, we have that $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_0})$ since $r^{-1}(b_{i0})$ is contractible in X_Γ. For $\Gamma_0 = \emptyset$, we have that $\pi_1(X_\Gamma)$ is nonabelian.

By induction, if $\Gamma_{n-1} \setminus C$ contains a black branch vertex of distance 1 to a terminal vertex we let $B_{n-1} = \{b_{n-1,1}, \ldots, b_{n-1,k_{n-1}}\}$ be the set of all outermost black branch vertices where each $b_{n-1,i}$ has distance 1 from a terminal vertex $w_{n-1,i}$. Let the component of $\Gamma_{n-1} \setminus (st(b_{n-1,i}) \cup w_{n-1,i})$ that is disjoint from C be denoted $T_{n-1,i}$. If there exists at least one component $T_{n-1,i}$ that is not 1-connected let $\Gamma_n = \emptyset$. If each $T_{n-1,i}$ is 1-connected then let $\Gamma'_n = \Gamma_{n-1} \setminus (\bigcup st(b_{n-1,i}) \cup \bigcup w_{n-1,i} \cup \bigcup T_{n-1,i})$. If Γ'_n is pruned the let $\Gamma_n = \Gamma'_n$, otherwise let Γ_n be the pruned Γ'_n.

We define the **core reduced graph** Γ_{CR} of Γ as follows:

$$\Gamma_{CR} = \begin{cases} \emptyset, & \text{if } \Gamma_n = \emptyset \text{ for some } n \geq 0, \text{otherwise} \\ \Gamma_n, & \text{for the smallest } n \text{ such that } \Gamma_n \text{ does not contain a black branch vertex of} \\ & \text{distance 1 to a terminal vertex} \end{cases}$$

For a core reduced graph Γ_{CR} of Γ where $\Gamma_{CR} \neq \emptyset$, we have that $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_{CR}})$. While if $\Gamma_{CR} = \emptyset$ then $\pi_1(X_\Gamma)$ is nonabelian.

Corollary 6.3.1. *Let Γ be a bicolored pruned trivalent graph such that X_Γ is a trivalent 2-stratifold that has fundamental group $\mathbb{Z} \times \mathbb{Z}_m$ for $m > 1$. Let Γ_{CR} be the core reduced graph of Γ. Then all of the following are satisfied.*

1. *$\Gamma_{CR} \neq \emptyset$;*
2. *The graph Γ_{CR} is homotopy equivalent to S^1 but not homeomorphic to S^1, all white vertices are genus 0 and all terminal vertices are white, and at least one black vertex belonging to the cycle C of Γ_{CR} is a branch vertex. Further all edges of Γ_{CR} incident to a terminal white vertex have label 2.*

Proof. Since $\pi(X_\Gamma)$ is abelian, $\Gamma_{CR} \neq \emptyset$. By corollary 6.1.4, the graph Γ_{CR} is homotopy equivalent to S^1 but not homeomorphic to S^1, all white vertices are genus 0 and all terminal vertices are white, and at least one black vertex belonging to the cycle C of Γ_{CR} is a branch vertex. The graph Γ_{CR} contains no terminal q-strings and no black branch vertex of distance 1 to a terminal vertex. If v is a white terminal vertex of Γ_{CR} then v is contained in a terminal p-string and the edge label incident to v is 2.

□

For a trivalent 2-stratifold with fundamental group $\mathbb{Z} \times \mathbb{Z}_m$, we show that $\mathbb{Z} \times \mathbb{Z}_m \cong \mathbb{Z} \times \mathbb{Z}_{2^k}$ as this will simplify our classification results.

Theorem 6.3.2. *Let Γ be a bicolored pruned trivalent graph. If $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}_m$ for $m > 1$ then $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}_{2^k}$ for $k > 0$.*

Proof. Let Γ_{CR} be the core reduced graph of Γ.

Suppose that $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}_m$ for $m > 1$. By corollary 6.3.1, $\Gamma_{CR} \neq \emptyset$ and the graph Γ_{CR} is homotopy equivalent to S^1 but not homeomorphic to S^1, all white vertices are genus 0 and all terminal vertices are white, and at least one black vertex belonging to the cycle C of Γ_{CR} is a branch vertex. Further, all edges of Γ_{CR} incident to a terminal white vertex have label 2.

By theorem 6.2.7, Γ_{CR} has all white vertices of degree ≤ 2, all black vertices of Γ_{CR} contained in the cycle C are branch vertices, and Γ_{CR} contains no horned trees. Let L be a linear subgraph of Γ_{CR} whose initial vertex is v_i'' and whose terminal vertex w is a white terminal vertex of Γ_{CR} and R_i where $L \cap C = v_i''$. Then by theorem 6.2.7, L is an O-string and $\pi_1(X_{\Gamma_{CR}}) \cong \mathbb{Z} \times \mathbb{Z}_{2^k}$ for $k > 0$ where Γ_{CR} is orientable if $k > 1$ otherwise Γ_{CR} is orientable or nonorientable. The integer $k \geq 1$ corresponds to the minimal number of edges with label 2 in all linear subgraphs L whose initial vertex is v_i'' and whose terminal vertex is a terminal vertex of Γ_{CR} where $L \cap C = \emptyset$ for $1 \leq i \leq n$.

□

We state the classification result.

Theorem 6.3.3. *Let Γ be a bicolored pruned trivalent graph. Then $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}_{2^k}$ for $k > 0$ if and only if the following hold:*

1. *Γ_X is homotopy equivalent to S^1 but not homeomorphic to S^1, all white vertices are genus 0, and all terminal vertices are white;*

2. *The core reduced graph $\Gamma_{CR} \neq \emptyset$ and all edges of Γ_{CR} incident to a terminal white vertex of genus 0 have label 2;*

3. *The graph Γ_{CR} is homotopy equivalent to S^1 but not homeomorphic to S^1, all white vertices are genus 0 and all terminal vertices are white, and at least one black vertex belonging to the cycle C of Γ_X is a branch vertex;*

4. *The graph Γ_{CR} contains no horned trees, all white vertices of Γ_{CR} are of degree ≤ 2, and all black vertices contained in C are branch vertices;*

5. *Let L be an linear subgraph of Γ_{CR} whose initial vertex is v_i'' and whose terminal vertex w is a white terminal vertex of Γ_{CR}. Then L is an O-string that contains $r \geq k$ edges with label 2 and there exists at least one L that contains k edges with label 2. If $k > 1$ then Γ_{CR} is orientable otherwise Γ_{CR} is either orientable or nonorientable.*

Proof. Suppose that $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}_{2^k}$. Then by the proof of theorem 6.3.2 the result holds.

Suppose that either conditions 1-5 holds. Then by the proof of theorem 6.2.7, $\pi_1(X_{\Gamma_{CR}}) \cong \mathbb{Z} \times \mathbb{Z}_{2^k}$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_{CR}})$. □

6.4 Trivalent 2-stratifolds with $\pi_1 = \mathbb{Z} \times \mathbb{Z}$

For a trivalent bicolored graph Γ, we now describe the necessary and sufficient conditions on Γ for $\pi_1(X_\Gamma)$ to be isomorphic to $\mathbb{Z} \times \mathbb{Z}$ where $\Gamma = \Gamma_X$.

Lemma 6.4.1. *Let Γ be a bicolored pruned trivalent graph. If $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}$ then the graph Γ_X is a tree, all terminal vertices are white, and contains one white vertex of genus 1 while all other white vertices are genus 0.*

Proof. Suppose that $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}$. By lemma 3.4.5, Γ_X is homotopy equivalent to S^1, all white vertices are genus 0, and all terminal vertices are white or Γ_X is a tree, all terminal vertices are white, and contains one white vertex of genus 1 while all other white vertices are genus 0.

Suppose Γ_X is homotopy equivalent to S^1, all white vertices are genus 0, and all terminal vertices are white. It follows by lemma 6.1.2 that Γ_X that is homotopy equivalent to S^1 but not homeomorphic to S^1 and at least one black vertex belonging to the cycle C of Γ_X is a branch vertex.

Since $\pi(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}$, $\Gamma_{CR} \neq \emptyset$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_{CR}})$. Then the graph Γ_{CR} is homotopy equivalent to S^1 but not homeomorphic to S^1, all white vertices are genus 0 and all terminal vertices are white, and at least one black vertex belonging to the cycle C of Γ_{CR} is a branch vertex. The set of black vertices on the cycle C of Γ_{CR} at distance 1 from a terminal vertex of Γ_{CR} is empty otherwise $\pi_1(X_{\Gamma_{CR}}) \cong \mathbb{Z}$. If v is a white terminal vertex of Γ_{CR} then v is contained in a terminal p-string and the edge label incident to v is 2.

By lemma 6.2.2 and the proof of lemma 6.2.6, all white vertices contained in Γ_{CR} are of degree ≤ 2, otherwise $\pi_1(X)$ is nonabelian. By lemma 6.2.2, Γ_{CR} contains no horned trees. Since $X_{\Gamma_{CR}}$ has abelian fundamental group then each C_i is a $L[p_i, q_i]$ graph. For each R_i, let $k_i > 0$ be the minimum number of edges with label 2 in all linear subgraphs L contained in R_i whose initial vertex is v_i'' and whose terminal vertex is a terminal vertex of Γ_X where $L \cap C = \emptyset$. If R_i is not a p-string of length $2k_i$ then replace R_i with a p-string of length $2k_i$. Let the resulting graph be called Γ'_{CR}. Then Γ'_{CR} is an echinus graph where $\Gamma'_{CR} = E[p_1, q_1, r_1; \ldots; p_n, q_n, r_n]$. Then $\pi_1(X_{\Gamma_{CR}}) \cong \pi_1(X_\Gamma)$ and $\pi_1(X_{\Gamma_{CR}}) \cong \pi_1(X_{\Gamma'_{CR}})$.

For $\Gamma'_{CR} = E[p_1, q_1, r_1; \ldots; p_n, q_n, r_n]$, if either $p_1 + \ldots + p_n = 0$ and $q_1 + \ldots + q_n \neq 0$ or $p_1 + \ldots + p_n \neq 0$ and $q_1 + \ldots + q_n = 0$ then $\pi_1(X_{\Gamma'_{CR}}) \cong \mathbb{Z}$ by proposition 5 of [10]. If $p_1 + \ldots + p_n \neq 0$

and $q_1 + \ldots + q_n \neq 0$ then $\pi_1(X_{\Gamma'_{CR}})$ is nonabelian by corollary 6.2.5. If $p_1 + \ldots + p_n = 0$ and $q_1 + \ldots + q_n = 0$ then $\pi_1(X_{\Gamma'_{CR}})$ is $\mathbb{Z} \times \mathbb{Z}_{2^k}$ for $k > 0$ or $\pi_1(X_{\Gamma'_{CR}})$ is nonabelian.

We conclude that Γ_X is a tree where all terminal vertices are white and Γ_X contains one white vertex of genus 1 while all other white vertices are genus 0.

□

It is assumed for the reminder of the section that all 2-stratifolds X have an associated graph Γ_X where Γ_X **is a tree, all terminal vertices are white, and contains one white vertex of genus** 1 **while all other white vertices are genus** 0**.**

Lemma 6.4.2. *Let X be a pruned trivalent 2-stratifold where the graph Γ_X has a label 2 for all edges incident to a terminal white vertex of genus 0. Then X has nonabelian fundamental group if Γ_X contains at least one of the following:*

1. *a white vertex of genus 1 and a white vertex of genus 0 with degree > 2;*
2. *a white vertex of genus 1 and a horned tree H_T;*
3. *a white vertex of genus 1 with degree ≥ 1;*

Proof. We assume that Γ_X is not a single white vertex of genus 1.

(1.) Let v be a white vertex of genus 1 and w be a white vertex of degree 3. Let L be the linear subgraph of Γ_X with terminal vertices v, w. Suppose e is the edge in L incident to w. Let P be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus e$ that contains w. If Γ_X is pruned at P, the resulting graph P' has a corresponding 2-stratifold $X_{P'}$ with nontrivial fundamental group $\pi_1(X_{P'})$ by Lemma 4.3.2. Now for the graph Γ_X, attach a white vertex of genus 0 with an edge of label 1 for all black vertices not contained in P. Then there is an epimorphism from $\pi_1(X) \to \pi_1(X_{P'}) \star \mathbb{Z} \times \mathbb{Z}$.

(2.) Suppose that v is a white vertex of genus 1 and H_T are disjoint. Attach to each black vertex not contained in H_T a white vertex of genus 0 with edge label 1. Then there is an epimorphism from $\pi_1(X) \to \mathbb{Z}_2 \star \mathbb{Z} \times \mathbb{Z}$.

(3.) Let v be the white vertex of genus 1. Suppose v has degree ≥ 2. Let E be all edges incident to v except for one edge e. Let R be the component of $\Gamma \setminus E$ that contains v. Pruning Γ at R results in a tree R' with a label 2 for all edges incident to a terminal white vertex of genus 0 and v is a white vertex of degree 1. We assume that v has degree 1 and all other white vertices of Γ_X are degree ≤ 2.

Suppose that Γ_X has no black vertices of degree 3. The vertex v is terminal and Γ_X is a linear graph. Orient the graph Γ_X so that vertices are ordered as $w_0 - b_1 - w_1 - b_2 - ... - b_r - w_r$ with corresponding edge labels $m_1 - n_1 - \ldots - m_r - n_r$ where $w_0 = v$ and $w_r = w$ where w is the other terminal vertex of Γ_X. By (2.), Γ_X contains no horned trees. Then either $m_1 = 2$, $n_1 = 1$ and $m_i = 1$, $n_i = 2$ for $2 \leq i \leq r$ or $m_i = 1$, $n_i = 2$ for $1 \leq i \leq r$.

Suppose that $m_1 = 2$, $n_1 = 1$ and $m_i = 1$, $n_i = 2$ for $2 \leq i \leq r$. Then prune Γ_X at the linear subgraph with initial vertex w_0 and terminal vertex w_2. Let the resulting graph be Γ'. Then $\pi_1(X_{\Gamma'})$ has the following presentation:

$$\{y_1, y_2, c, b_1 | cy_1y_2y_1^{-1}y_2^{-1}, c = b_1^2, b_1^2\}.$$

This presentation is equivalent to:

$$\{y_1, y_2, b_1 | y_1y_2y_1^{-1}y_2^{-1}, b_1^2\}.$$

Suppose that $m_i = 1$, $n_i = 2$ for $1 \leq i \leq r$. Then prune Γ_X at the linear subgraph with initial vertex w_0 and terminal vertex w_1. Let the resulting graph be Γ'. Then $\pi_1(X_{\Gamma'})$ has the following presentation:

$$\{y_1, y_2, c, b_1 | cy_1y_2y_1^{-1}y_2^{-1}, c = b_1, b_1^2\}.$$

This presentation is equivalent to the following:

$$\{y_1, y_2 | [y_1, y_2]^2\}.$$

Suppose that Γ_X has $k > 0$ black vertices of degree 3. Let b be a black vertex of degree 3 where b is adjacent to the vertices v_1, v_2, v_3 such that v_j is the initial vertex of a terminal linear subgraph T_j for $i = 1, 2$. If T_j contains a horned tree then X_Γ has nonabelian fundamental group. If v is contained in a terminal linear subgraph T_1 or T_2 of Γ_X then there exists another black branch vertex b' such that b' is adjacent to the initial vertex of terminal linear subgraphs T_1', T_2'. The subgraphs T_1', T_2' are p-strings. We assume that T_1, T_2 are p-strings. Apply operation $B1$ on $st(b) \cup T_1 \cup T_2$.

The resulting graph Γ' contains with $k-1$ black vertices of degree 3 and $\pi_1(X_{\Gamma'}) \cong \pi_1(X_{\Gamma'})$. By induction hypothesis, the result holds.

□

Corollary 6.4.3. *Let Γ be a bicolored pruned trivalent graph. If $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}$ then the core reduced graph $\Gamma_C \neq \emptyset$, Γ_C is a single white vertex of genus 1 with no edges, and X_{Γ_C} is a 2-torus.*

Proof. Since $\pi(X_\Gamma)$ is abelian, $\Gamma_C \neq \emptyset$. Then the graph Γ_C is a tree, all terminal vertices are white, and contains one white vertex of genus 1 while all other white vertices are genus 0.

Suppose that Γ_C is not a single white vertex of genus 1. Then let v be a white terminal vertex of genus 0. The graph Γ_C contains no terminal q-strings and no black branch vertex of distance 1 to a terminal vertex. Then v is contained in a terminal p-string and the edge label incident to v is 2. By lemma 6.4.2, then X_{Γ_C} has nonabelian fundamental group. Therefore Γ_C consists of a single white vertex of genus 1. □

Theorem 6.4.4. *Let Γ be a bicolored pruned trivalent graph. Then $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}$ if and only if the following hold:*

1. *The graph Γ_X is a tree, all terminal vertices are white, and contains one white vertex of genus 1 while all other white vertices are genus 0.*
2. *The core reduced graph $\Gamma_C \neq \emptyset$, Γ_C is a single white vertex of genus 1 with no edges, and X_{Γ_C} is a 2-torus.*

Proof. Suppose that $\pi_1(X_\Gamma) \cong \mathbb{Z} \times \mathbb{Z}$. Then (1.) follows by lemma 6.4.1 and (2.) follows by corollary 6.4.3.

Suppose that conditions 1-2 holds. Then $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$.

□

BIBLIOGRAPHY

[1] Hyman Bass, Covering theory for graphs of groups, Journal of Pure and Applied Algebra, Volume 89, Issues 1–2, 1993, Pages 3-47, ISSN 0022-4049, https://doi.org/10.1016/0022-4049(93)90085-8. (http://www.sciencedirect.com/science/article/pii/0022404993900858)

[2] J.S. Carter, Reidemeister/Roseman-type moves to embedded foams in 4-dimensional space, Series on Knots and Everything 56, New Ideas in Low Dimensional Topology, 1-30 (2015).

[3] K. Eto, S. Matsuzaki, M. Ozawa, An obstruction to embedding 2-dimensional complexes into the 3-sphere, Topology and its Appl. 198 (2016) 117–125.

[4] S. Friedl, T. Kitayama, M. Nagel, A note on the existence of essential tribranched surfaces, Topology and its Applications, Volume 225, 2017, Pages 75-82,ISSN 0166-8641

[5] J.C. Gómez-Larrañaga, F. González-Acuña, Wolfgang Heil, 2-stratifolds, in “A Mathematical Tribute to José María Montesinos Amilibia", Universidad Complutense de Madrid, 395-405 (2016).

[6] J.C. Gómez-Larrañaga, F. González-Acuña, Wolfgang Heil, 2-dimensional stratifolds homotopy equivalent to S^2, Topology Appl. 209, 56-62 (2016).

[7] J.C. Gómez-Larrañaga, F. González-Acuña, Wolfgang Heil, Classification of Simply-connected Trivalent 2-dimensional Stratifolds, Top. Proc. (2018)

[8] J.C. Gómez-Larrañaga, F. González-Acuña, Wolfgang Heil, 2-stratifold groups have solvable Word Problem, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, Online First Articles ISSN: 1578-7303 (Print) 1579-1505 (Online)

[9] J.C. Gómez-Larrañaga, F. González-Acuña, Wolfgang Heil, 2-stratifold spines of closed 3-manifolds, arXiv:1707.05663 (2017).

[10] J.C. Gómez-Larrañaga, F. González-Acuña, Wolfgang Heil, 2-stratifolds with fundamental group $\mathbb{Z}$, arXiv:1812.01589v1 (2018).

[11] K. Ishihara, Y. Koda, M. Ozawa, K. Shimokawa, Neighborhood equivalence for multibranched surfaces in 3-manifolds, Topol. Appl. 257 (2019), 11–21.

[12] S. Katok, Fuchsian groups, Chicago lectures in mathematics series, Chicago : University of Chicago Press, 1992

[13] R. C. Lyndon and P. E. Schupp, Combinatorial Group Theory, Modern Surveys in Math., no. 89, Springer Verlag, Berlin, 1977.

[14] W. Magnus, Noneuclidean tesselations and their groups, Pure and applied mathematics (Academic Press), 61. , New York, Academic Press, 1974

[15] S. Matsuzaki and M. Ozawa, Genera and minors of multibranched surfaces, Topology and its Applications 230, 621-638 (2017).

[16] M. Ozawa, A partial order on multibranched surfaces in 3-manifolds, arXiv: 1905.01055 (2019)

[17] J. Serre, Trees, Springer-Verlag, 1980.

BIOGRAPHICAL SKETCH

John Henry Bergschneider was born on January 27th, 1992 in Columbus, Georgia. He received a Bachelor of Science with a major in mathematics from Florida State University in May 2014. In August 2014, he entered the mathematics graduate program at Florida State University where he received his Ph.D. under the direction of Wolfgang Heil.