CLASSIFICATION OF TRIVALENT 2-STRATIFOLDS WITH FINITE FUNDAMENTAL GROUP

A PREPRINT

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September 4, 2021

ABSTRACT

A 2-stratifold is a compact topological space such that each point has a neighborhood homeomorphic where n-sheets meet. These spaces are a generalization of 2-manifolds, however there is no complete classification of 2-stratifolds. In this paper, we determine the finite groups that arise as the fundamental group of a 2-stratifold. Trivalent 2-stratifolds are a subclass locally modelled on where 3-sheets meet. We then give a classification of trivalent 2-stratifold with finite fundamental group.

1 Introduction

A **2-stratifold** X is a compact, Hausdorff space X that contains a closed (possibly disconnected) 1-manifold $X^{(1)}$ as a closed subspace with the following property: Each point $x \in X^{(1)}$ has a neighborhood homeomorphic to $\mathbb{R} \times CF$, where CF is the open cone on the finite set F with cardinality > 2, and where $X \setminus X^{(1)}$ is a (possibly disconnected) 2-manifold. These spaces appeared while studying Lusternick-Schnirelman type decompositions of 3-manifold in [1]. Related stratified spaces called multibranched surfaces arose while studying the embeddability of 2-dimensional cell complexes into the 3-sphere. An obstruction for embedding a multibranched surface into the 3-sphere was given in [2]. Then embeddings of multibranched surfaces in 3-manifolds are studied in [3],[4], and [5].

If each point of $X^{(1)}$ has a neighborhood where 3 sheets meet then X is called trivalent. Trivalent 2-stratifolds are a subset of spaces called foams. Foams and 2-stratifolds appear as spines of 3-manifolds. While all special spines are foams, very few 2-stratifolds occur as spines of 3-manifolds. Spines of closed 3-manifolds that are 2-stratifolds have been classified in [6]. It was shown in [5] that every multibranched surface, and hence every 2-stratifold, embedds in $\mathbb{R}^4$. Reidemeister/Roseman-type moves on knotted foams in $\mathbb{R}^4$ have been studied in [7].

Any F-group can be realized as the fundamental group of a 2-stratifold. This family of groups are essentially the fuchsian groups. In general the fundamental group of a 2-stratifold can be represented as the fundamental group of a certain type of graph of groups. However these spaces are not determined by their fundamental group and there is no classification of general 2-stratifolds. For 1-connected trivalent 2-stratifolds a classification was given in [8]. Then a classification of trivalent 2-stratifolds with fundamental group $\mathbb{Z}$ followed in [9]. Since the homeomophism class of a 2-stratifold is determined by a bicoloured labelled graph Γ_X, these classifications are in terms of conditions that can be read off the graph Γ_X.

We extend the classification to trivalent 2-stratifolds with finite fundamental group in this paper. This classification is given by Theorems 7.3-7.7. The main step in proving the classification is to determine the finite fundamental groups of a trivalent 2-stratifold. This is given by the following:

Theorem 7.2 *Let X_Γ be a trivalent 2-stratifold. If X_Γ has finite fundamental group then $\pi_1(X_\Gamma)$ is isomorphic to either $\mathbb{Z}_{2^{k+1}}$, $\mathbb{Z}_{3*2^k}$, or the dihedral group $D_{2^{k+1}}$ where $k \geq 0$.*

The outline of the paper is as follows. In section 3, we prove that the finite fundamental groups of 2-stratifolds are the finite F-groups. Then in section 4 and section 5, we find necessary conditions and sufficient conditions for a trivalent 2-stratifold X_Γ to have finite fundamental group. To find these conditions we introduce a surgery type move on the graph Γ_X called operation $B1$. In the final section, we produce the classification of trivalent 2-stratifolds with finite fundamental group.

Acknowledgments

First, I would like to thank my thesis advisor Wolfgang Heil for his helpful advice and illustrations throughout this project. I am also appreciative of Aamir Rasheed for his many discussions on HNN extensions and amalgamated free products. I would also like to thank Michael Niemeier for his feedback on group computations. Finally, I would like to thank Opal Graham for her suggestions throughout the life cycle of this project.

2 Properties of 2-stratifolds

We find necessary conditions for a 2-stratifold X to have finite fundamental group in this section. This is done in Lemma 2.2. Beforehand we review how to obtain an associated bipartite labelled graph Γ_X for X.

A component B of $X^{(1)}$ has a regular neighborhood denoted by $N(B) = N_\pi(B)$. The regular neighborhood $N_\pi(B)$ is homeomorphic to the mapping cylinder of f where if π is the partition $n_1 + n_2 + \ldots + n_r$ of d, the map $f : \tilde{B} \to B$ is from a closed 1-manifold with components $\tilde{B}_1$, $\tilde{B}_2$,..., $\tilde{B}_r$ and the restriction of f to $\tilde{B}_i$ is an n_i-fold covering $1 \leq i \leq r$. The space $N_\pi(B)$ is determined by the partition of d.

For a 2-stratifold X there is an associated bipartite graph Γ_X embedded in X. For disjoint components B and B' of $X^{(1)}$ allow $N(B)$ and $N(B')$ be chosen sufficiently small so that $N(B)$ and $N(B')$ are disjoint. The white vertices w_i of the graph Γ_X are the components W_i of $M = \overline{X \setminus \cup_i N(B_i)}$ for all components B_i of $X^{(1)}$. The black vertices b_i of graph Γ_X correspond to the regular neighborhood $N(B_i)$. An edge is e_{ij} is component of E_{ij} of ∂M that joins b_j and w_i if $W_j \cap N(B_i) = E_{ij}$. We label the white vertices w_i of graph Γ_X with the genus of the corresponding surface W_i. By convention, we assign a negative genus g to a nonorientable surface. Each edge of Γ_X is labeled by an integer k, where k is the summand of the partition π corresponding to the boundary component E of $N(B_i)$.

Notation 2.1. *The labelled bipartite graph associated to a 2-stratifold X is denoted by Γ_X and X is denoted by X_Γ.*

For a given labelled graph Γ, by pruning away edges and vertices we obtain a subgraph Γ' such that there is an epimorphism from $\pi_1(X_\Gamma)$ to $\pi_1(X_{\Gamma'})$. It was shown in [10], there is a retraction $r : X \to \Gamma_X$ such that $r^{-1}(b)$ is a singular curve $B \in X^{(1)}$ and $r^{-1}(w)$ is a 2-manifold W. Let Γ_0 be a subgraph of Γ_X and let $Y = r^{-1}(\Gamma_0)$. The space Y contains boundary curves corresponding to $St(\Gamma_0) - \Gamma_0$, where $St(\Gamma_0)$ is the closed star of Γ_0 in Γ_X. Denote the labelled edges of $St(\Gamma_0) - \Gamma_0$ adjacent to a black vertex of Γ_X as E. Attach disks with a degree 1 attaching maps to the boundary curves of Y. The resulting space is a 2-stratifold $Y' = X_{\Gamma'}$ where Γ' is obtained by deleting the complement of $\Gamma_0 \cup E$ from Γ_X then attaching white vertices of genus zero to the labelled edges of E. We say Γ' is obtained from Γ by pruning at Γ_0.

If Γ is a bipartite labelled tree then there is a unique 2-stratifold X such that $\Gamma_X = \Gamma$. We now give necessary conditions on Γ_X for X to have finite fundamental group.

Lemma 2.2. *Let X be 2-stratifold with graph Γ_X. If $\pi_1(X)$ is finite then Γ_X is a tree that satifies one of the following set of conditions:*

1. *Γ_X has all white vertices of genus 0, one black terminal vertex and all other terminal vertices are white.*
2. *Γ_X has at most one white vertex of genus -1 while all other white vertices are genus 0, and all terminal vertices are white.*

Proof. The retraction $r : X \to \Gamma_X$ induces an epimorphism $r_\star : \pi_1(X) \to \pi_1(\Gamma_X)$. Therefore Γ_X is a tree. If w is a white vertex of Γ_X then pruning Γ_X at w results in a closed 2-manifold W' with finite fundamental group. The 2-manifold W' is either a 2-sphere or real projective plane. It was shown in [9] that $\pi_1(X)$ is infinite if Γ_X contains at either two black terminal vertices, two white vertices of genus g, or a black terminal vertex and white vertices of genus g for $g \neq 0$. Therefore Γ_X contains at most one white vertex of genus -1 or one black terminal vertex. If Γ_X contains one black terminal vertex then all other terminal vertices are white and all white vertices are genus zero. If Γ_X contains a white vertex of genus -1 then all other white vertices are genus zero and all terminal vertices are white. ∎

3 Finite 2-stratifolds groups

We determine the finite fundamental groups of 2-stratifolds in this section. This is given by Theorem 3.6. To find these finite groups, we represent the fundamental group of X as a fundamental group of a graph of groups and show the reduced graph of groups must be a vertex.

An **abstract graph** Y consists of two sets: $V = V(Y)$, vertices, and $E = E(Y)$, (oriented) edges, together with maps $E \to V \times V$, $e \to (o(e), t(e))$ (the originating and terminal vertices of e), and $E \to E$, $e \to \bar{e}$ (reversal of orientation) such that $e = \bar{\bar{e}}$, $e \neq \bar{e}$, $t(e) = o(\bar{e})$, and $o(e) = t(\bar{e})$. A **graph of groups** (G, Y) consists of an abstract graph Y, two families of groups $\{G_v | v \in V(Y)\}$, $\{G_e | e \in E(Y)\}$ such that $G_e = G_{\bar{e}}$, and a family of monomorphisms $\{f_e\}$ with $f_e : G_e \to G_{t(e)}$, $f_{\bar{e}} : G_{\bar{e}} \to G_{o(e)}$. For a graph of groups (G, Y), the group $F(G, Y)$ is generated by the vertex groups G_v and elements e corresponding to the elements of $E(Y)$, subject to the relations $\bar{e} = e^{-1}$ and $e f_e(x) e^{-1} = f_{\bar{e}}(x)$ for all $x \in G_e$ and for each $e \in E(Y)$. For a fixed vertex v_0, the **fundamental group** $\pi_1(G, Y, v_0)$ **of the graph of groups** (G, Y) is the subgroup of $F(G, Y)$ generated by all words

$$w = r_0 e_1 r_1 e_2 \dots e_n r_n$$

where $v_0 - v_1 - v_2 - \dots - v_n$ is an edge path with initial and terminal vertex $v_0 = v_n$ (i.e. a cycle based at v_0), successive edges e_i (joining v_{i-1} to v_i) and $r_i \in G_{v_i}$. The word $w = r_0 e_1 \dots e_n r_n$ of length n is **reduced**, if for $n = 0$, $r_0 \neq 1$; for $n \geq 1$, $r_i \notin f_e(G_{e_i})$, for each index i such that $e_{i+1} = \bar{e}_i$. The group $\pi_1(G, Y, v_0)$ is independent of the choice of v_0.

Serre showed the following in [11]

Lemma 3.1. *If $w \in \pi_1(G, Y, v_0)$ is a reduced word then $w \neq 1$ in $\pi_1(G, Y, v_0)$. If (G, Y) is a graph of groups, the homomorphism $G_v \to \pi_1(G, Y, v_0)$ is injective.*

A **subgraph of subgroups** (G', Y') of (G, Y) is a graph of groups where Y' is a connected subgraph of Y, $G'_v \leq G_v$ for all v in Y', and for all $e \in E(Y')$, $G'_e \leq G_e$ and $f'_e = f_e|_{G'_e}$. Bass proved the next lemma in [12].

Lemma 3.2. *If (G', Y') is a subgraph of groups of (G, Y), then the natural homomorphism $i_\star : \pi_1(G', Y', v_0) \to \pi_1(G', Y', v_0)$ is injective.*

We will denote the fundamental group $\pi_1(G, Y, v_0)$ as $G_{v_0} \star_{G_e} G_{v_1}$ if the graph of groups (G, Y) has a graph Y with one edge $\{e, \bar{e}\}$ and two vertices v_0, v_1.

Lemma 3.3. *Let (G, Y) be a graph of groups where $G = \pi_1(G, Y, v_0)$. Let $\{e, \bar{e}\}$ be an edge contained in Y. If $o(e) \neq t(e)$, $f_e, f_{\bar{e}}$ are not surjective, and $G_{o(e)}$, $G_{t(e)}$ are nontrivial then G is not finite and not abelian.*

Proof. We write $f_e, f_{\bar{e}}$ as inclusions so that $G_e < G_{v_1}$, $G_{\bar{e}} < G_{v_0}$.

(1.) Let $v_0 = o(e)$ and $v_1 = t(e)$. Let (H, X) be a subgraph of subgroups (G, Y) where $H_v = G_v$ for all $v \in V(X)$, $H_e = G_e$ for all $e \in E(X)$, and X consists of two vertices v_0, v_1 and a single edge $\{e, \bar{e}\}$. The fundamental group $\pi_1(H, X, v_0) = N$ is a subgroup of G. The group N is the free product with amalgamation $G_{v_0} \star_{G_e} G_{v_1}$. There exists $a \in G_{v_0}$ and $b \in G_{v_1}$ such that $a \notin G_{\bar{e}}$ and $b \notin G_e$. The word $(ab)^k$ is a reduced word in N for all k and by lemma 3.1 $(ab)^k \neq 1$ in N. The word ab has infinite order. The word $aba^{-1}b^{-1}$ is a reduced word in N and $aba^{-1}b^{-1} \neq 1$ in N.

□

An edge e of a graph of groups (G, Y) is said to be **trivial** if $o(e) \neq t(e)$ and f_e is an isomorphism. An edge e of a graph of groups (G, Y) where $G_{t(e)} = \{\emptyset\}$ and $o(e) \neq t(e)$ is trivial by this definition. **Collapsing a trivial edge** e of a graph of groups (G, Y) is the process constructing a new graph of groups (G', Y') where Y' is obtained from Y by contracting $\{e, \bar{e}\}$ to a point E, set $G_E := G_{o(e)}$, and $G' = G$ on all remaining edges and vertices. The fundamental group of (G', Y') is isomorphic to the fundamental group of (G, Y). A graph of groups with no trivial edge is said to be **reduced**.

Let Y be an abstract graph. The realization of Y is the topological graph **Y** with vertices $v(Y)$ and edges corresponding to the edges $\{e, \bar{e}\}$.

Lemma 3.4. *Let (G, Y) be a graph of groups with a finite graph Y. If (G, Y) is a graph of groups where $\pi_1(G, Y, v_0)$ is finite then $\pi_1(G, Y, v_0) \cong \pi_1(G', Y', v'_0)$ such that (G', Y') is a reduced graph of groups where the graph Y' is a vertex v'_0 with no edges and the vertex group $G_{v'_0}$ of (G', Y') is isomorphic to a vertex group G_w of (G, Y).*

Proof. Let **Y** be the realization of Y. For any graph of groups (G, Y) there is a surjective homomorphism $\pi_1(G, Y, v_0) \to \pi_1(\mathbf{Y}, v_0)$ where $\pi_1(\mathbf{Y}, v_0)$ is the fundamental group of the graph **Y**. If (G, Y) is a graph of groups where $\pi_1(G, Y, v_0)$ is finite then **Y** is a tree.

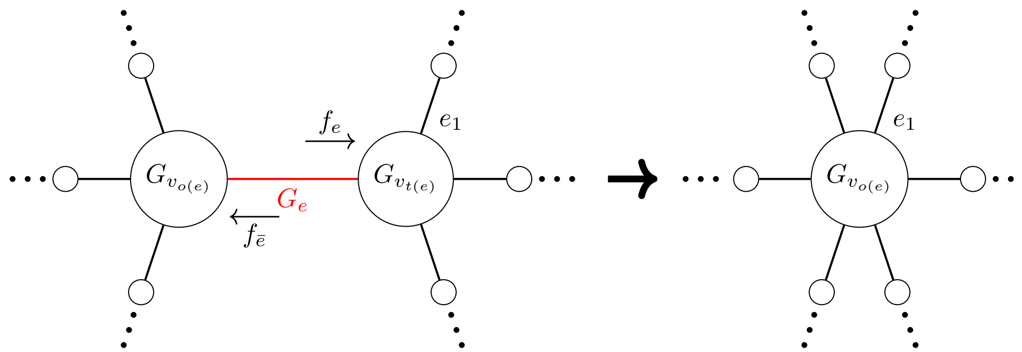

Figure 1: Collapsing a trivial edge.

For a graph of groups (G, Y) where the graph Y contains a single vertex, the graph Y must contain no edges by the previous paragraph.

Otherwise, by induction, we assume that for a graph of groups (G, Y) where $\pi_1(G, Y, v_0)$ is finite and Y contains $n-1$ vertices then $\pi_1(G, Y, v_0) \cong \pi_1(G', Y', v_0')$ where (G', Y') is a reduced graph of groups such that Y' is a vertex v_0' and the vertex group $G_{v_0'}$ of (G', Y') is isomorphic to a vertex group G_w of (G, Y).

Suppose that (G, Y) is a graph of groups where $\pi_1(G, Y, v_0)$ is finite and Y contains n vertices. Let (H, X) be a subgraph of subgroups (G, Y) where $H_v = G_v$ for all $v \in V(X)$, $H_e = G_e$ for all $e \in E(X)$, and X consists of two vertices v_1, v_2 and a single edge $\{e\}$ incident to v_1, v_2. Let $v_1 = o(e)$ and $v_2 = t(e)$. If $\{e, \bar{e}\}$ are nontrivial edges in (G, Y), then the fundamental group $\pi_1(H, X, v_1)$ is $G_{v_1} \star_{G_e} G_{v_2}$, which is infinite by lemma 3.3. But $\pi_1(H, X, v_1)$ is a subgroup of $\pi_1(G, Y, v_1)$ and every subgroup of a finite group is finite. At least one edge e' of $\{e, \bar{e}\}$ is trivial in (G, Y). Let (G', Y') be the graph of groups obtained by collapsing the trivial edge e' of the graph of groups (G, Y). In (G', Y'), Y' contains $n-1$ vertices.

□

For a 2-stratifold X_Γ, it was shown in [6] that $\pi_1(X_\Gamma)$ determines a graph of groups (G, Y) where $\mathbf{Y} = \Gamma_X$ such that $\mathbf{Y}$ is the realization of Y and $\pi_1(G, Y, v_0) \cong \pi_1(X_\Gamma)$. The graph Y is a bipartite graph which is induced by Γ_X. The groups G_b of the black vertices and the groups G_e of the edges are cyclic. The groups G_w of the white vertices with edges $e_1, \ldots, e_p$ labelled $m_1, \ldots, m_p$ have the following presentation,

$$G_w = \{c_1, \ldots, c_p, y_1, \ldots, y_n : c_1 \ldots c_p q = 1,\ c_1^{m_1}, \ldots, c_r^{m_r}\ (r \leq p)\},$$

where $p, n \geq 0$ and $q = [y_1, y_2] \ldots [y_{2g-1}, y_{2g}]$ or $q = y_1^2 \ldots y_g^2$. If a group G has a presentation given by G_w where all $m_i \geq 2$ and $r = p$ then G is an F**-group**. Otherwise G_w is a free product of cyclic groups.

The finite F-groups are determined in [13].

Lemma 3.5. *The group $\mathcal{F}$ is finite cyclic if and only if $n = 0$ and $p \leq 2$ or $n = 1$ and $p \leq 1$. The group $\mathcal{F}$ is finite non-cyclic if and only if $n = 0$, $p = 3$, and (m_1, m_2, m_3) is either $(2, 2, m)$ with $m \geq 2$ (dihedral group of order $2m$) or $(2, 3, k)$ with $3 \leq k \leq 5$ (the tetrahedral, octahedral, dodecahedral groups).*

Theorem 3.6. *Let X be a 2-stratifold. If X has finite fundamental group then $\pi_1(X)$ is either trivial, finite cyclic, dihedral group of order $2m$, or the tetrahedral, octahedral, dodecahedral groups.*

Proof. Suppose that (G, Y) is the associated graph of groups to $\pi_1(X_\Gamma)$ such that $\pi_1(G, Y, v_0) \cong \pi_1(X_\Gamma)$. If (G, Y) is a graph of groups where $\pi_1(G, Y, v_0)$ is finite then $\mathbf{Y}$ is a tree, all vertex groups G_v and all edge groups G_e are finite. The vertex groups G_w of (G, Y) are finite F-groups. The vertex groups G_b and edge groups G_e of (G, Y) are finite cyclic groups. By lemma 3.4, $\pi_1(G, Y, v_0)$ is isomorphic to a vertex group of (G, Y). Therefore $\pi_1(G, Y, v_0)$ is isomorphic to either the trivial group or a finite F-group.

□

4 Operation B1 on Trivalent 2-stratifolds

We first review the definition of a trivalent 2-stratifold X and other relevant definitions. Then we introduce a surgery type move on the graph Γ_X called operation $B1$. The section is completed by Corollary 4.3.1 which states if Γ_X is a tree then Γ_X contains a certain number of black vertices.

A 2-stratifold X is called **trivalent** if the graph Γ_X has every black vertex b either incident to three edges, each with label 1, two edges, one with label 1, the other with label 2, or b is a terminal vertex with adjacent edge of label 3. A graph Γ_X is also said to be **trivalent** if X_Γ is a trivalent 2-stratifold.

A p**-string** of length $2r$ is an oriented linear graph $w_0 - b_1 - w_1 - b_2 - ... - b_r - w_r$ with all white vertices w_i of genus 0, successive edge labels 1212...12 (starting at w_0) and with r labels of 2. A q**-string** is an oriented linear graph with all white vertices w_i of genus 0, successive edge labels 2121...21 (starting at w_0), and with r labels of 2. A p-string (or q-string) is **terminal** if w_r is a terminal white vertex of Γ. If L is a terminal q-string then pruning L from Γ_X does not alter the fundamental group of a X. A trivalent 2-stratifold graph Γ is **pruned** if Γ contains no terminal q-strings. A trivalent 2-stratifold X is also said to be **pruned** if the associated labeled graph Γ_X is pruned.

A linear bipartite labelled graph L with successive vertices $w_0 - b_1 - w_1 - \ldots - b_r - w_r$, successive labels $m_1, n_1, \ldots, m_r, n_r$ where m_i (resp. n_i) is the label of the edge joining b_i to w_{i-1} (resp. w_i) for $r = 1, \ldots, r$ will be denoted by $L = L(m_1, n_1, \ldots, m_r, n_r)$. A linear subgraph $L(m_1, n_1, \ldots, m_r, n_r)$ of Γ_X (resp. $L(n_1, \ldots, m_r, n_r)$) will be called **terminal** if w_r is a terminal vertex of Γ and vertices b_i, w_i for $i > 0$ (resp. b_{i+1}, w_i for $i > 0$) are of degree < 3. Let $L = L(m_1, n_1, \ldots, m_r, n_r)$ be a terminal linear subgraph of Γ where the initial vertex w_0 has genus g and all other white vertices in L have genus 0. Let $L(1, n_1 \ldots n_r)$ be a linear graph whose initial vertex has genus g while all other vertices have genus 0. L**-pruning** Γ **at** $L(m_1, n_1, \ldots, m_r, n_r)$ is the process of replacing $L(m_1, n_1, \ldots, m_r, n_r)$ with $L(1, n_1 \ldots n_r)$. In [14] it was shown, if $gcd(m_i, n_j) = 1$ for $1 \leq i \leq j \leq r$ then $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$.

For trivalent 2-stratifolds X whose graph Γ_X contains $n > 1$ black vertices of degree 3, the operation $B1$, (seen below), applied to the graph Γ_X produces a new graph Γ' that contains $n - 1$ black vertices of degree 3.

Let Γ be a trivalent graph containing a black vertex b of degree 3 with adjacent vertices v_1, v_2, v_3, such that v_i is the initial vertex of a terminal p-string P_i of length $2p_i$ for $i = 1, 2$. **Operation B1** produces a trivalent graph Γ' from Γ by replacing $st(b) \cup P_1 \cup P_2$ with a p-string P' (with initial vertex v_3) of length min$\{2p_1, 2p_2\}$. The p-string P' in Γ' will be referred to as the **associated** p**-string**.

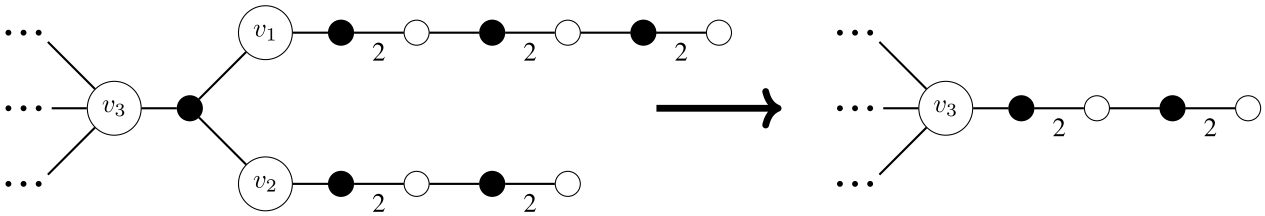

Figure 2: Operation B1

Lemma 4.1. *Let X be a trivalent 2-stratifold whose graph Γ_X contains $n > 1$ black vertices of degree 3. Let b to be a black vertex of Γ_X with degree 3 that is adjacent to the initial vertex of two terminal p-strings P_1, P_2 with length $2p_1, 2p_2$ respectively. Let Γ' be obtained from Γ by operation $B1$. Then $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$ and Γ' contains $n - 1$ black vertices of degree 3.*

Proof. L-prune the terminal p-strings P_i. In the resulting graph Γ', the black vertex b is adjacent to two terminal vertices v_1', v_2' where the edge incident to b and v_i' has label 2^{p_i}. L-pruning induces an isomorphism, so $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$. Let the terminal linear graph, whose initial vertex is b and whose terminal vertex is v_i', be called L_i. Construct Γ'' by replacing $(L_1 \setminus b) \cup (L_2 \setminus b)$ with a single terminal linear branch L'' of length 1, with initial vertex b, terminal vertex w of genus 0, and with edge label $min(2^{p_1}, 2^{p_2})$. The group $\pi_1(X_{\Gamma'})$ is isomorphic to $\pi_1(X_{\Gamma''})$. The stratifold $X_{\Gamma''}$ is not a trivalent 2-stratifold. Replace the terminal linear graph $L'' \cup st(b) \cup v_3$ with a p-string P' of length $min(2p_1, 2p_2)$ with initial vertex which has been replaced by v_3. The resulting graph Γ''' contains $n - 1$ black vertices of degree 3, $X_{\Gamma'''}$ is a trivalent 2-stratifold, and the fundamental group $\pi_1(X_{\Gamma'''})$ is isomorphic to $\pi_1(X_\Gamma)$. $\square$

Remark 4.2. *We note the operation $B1$ does not alter $\Gamma_X \setminus (st(b) \cup T_1 \cup T_2)$. Then $\Gamma' \setminus (P' \setminus v_3) = \Gamma_X \setminus (st(b) \cup P_1 \cup P_2)$. If S is a subgraph of Γ' that is contained in $\Gamma' \setminus (P' \setminus v_3)$ then the same subgraph in Γ_X contained* *in $\Gamma_X \setminus (st(b) \cup P_1 \cup P_2)$ will also be called S and vice versa. Whether S is a subgraph of Γ' or a subgraph of Γ_X will be determined by context.*

By inductively applying the operation $B1$, it will be shown that a trivalent 2-stratifold graph Γ_X will be produced with no black vertices of degree 3 if X has finite fundamental group. To insure this can be inductively done, we show that certain trivalent 2-stratifold graphs Γ_X have the property given in Corollary 4.3.1.

Lemma 4.3. *Suppose that Γ is a tree. If every nonterminal vertex of Γ has degree 3 then Γ contains two more terminal vertices than nonterminal vertices.*

Proof. Suppose the graph Γ has m total vertices then the number of edges is $m-1$ since Γ is a tree. If Γ contains k terminal vertices then the number of nonterminal vertices is $m-k$. By the handshaking lemma we have $k+3(m-k) = 2(m-1)$. The total number of vertices is then $m = 2k-2$. Therefore we get $(m-k) = k-2$. □

Corollary 4.3.1. *Let X be a trivalent 2-stratifold where Γ_X is a tree that contains $n > 1$ black vertices of degree 3 and all white vertices are degree ≤ 2. If Γ_X contains at most one black terminal vertex then Γ_X contains at least two black vertices of degree 3 that are adjacent to the initial vertex of two terminal linear subgraphs.*

5 Graphs of Trivalent 2-Stratifolds with Finite Fundamental Group

The goal of this section is to find further necessary conditions for a trivalent 2-stratifold to have finite fundamental group. These conditions are given by theorem 5.4. The following lemma was shown in [10] and will be used frequently.

Lemma 5.1. *Let X be a pruned trivalent 2-stratifold. If Γ_X has all white vertices of genus 0, all terminal edges have label 2, and all terminal vertices are white then X_Γ is not simply connected.*

In this section, we assume that all 2-stratifolds X satisfy a set of necessary conditions from lemma 2.2. Namely, the graph Γ_X is a tree that satisfies one of the following conditions: the graph Γ_X has exactly one black terminal vertex, all other terminal vertices are white, and all white vertices are genus 0; the graph Γ_X has exactly one white vertex of genus -1, all other white vertices are genus 0, and all terminal vertices are white; or the graph Γ_X has all white terminal vertices and white vertices are of genus 0.

We denote a linear subgraph L of Γ_X with vertices $w_0 - b_1 - w_1 - b_2 - w_2$, successive edge labels $2, 1, 1, 2$, and all white vertices w_i are of genus 0 as $L(2, 1, 1, 2)$.

Lemma 5.2. *Let X be a pruned trivalent 2-stratifold where the graph Γ_X has a label 2 for all edges incident to a terminal white vertex of genus 0. Then X has infinite fundamental group if Γ_X contains at least one of the following:*

1. *two linear subgraphs $L_1(2, 1, 1, 2)$ and $L_2(2, 1, 1, 2)$ where L_1 and L_2 are disjoint or L_1 and L_2 intersect at a vertex v such that v is a terminal vertex of L_1 and L_2;*
2. *a black terminal vertex with edge label 3 and a white vertex of degree > 2;*
3. *a white vertex of genus -1 and a white vertex of degree > 2;*
4. *a white vertex of genus -1 with degree ≥ 2;*
5. *or at least two white vertex w_1, w_2 of degree > 2.*

Proof. (1.) Attach to each black vertex not contained in $L_1(2, 1, 1, 2)$ or $L_2(2, 1, 1, 2)$ of Γ_X a white vertex of genus 0 with edge label 1. Then there is an epimorphism from $\pi_1(X) \to \mathbb{Z}_2 \star \mathbb{Z}_2$.

(2.) Assume that b is the black terminal vertex of Γ_X and w is the white vertex of degree > 2. Let L be the linear subgraph of Γ_X with terminal vertices b, w. Suppose e is the edge in L incident to w. Let P be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus e$ that contains $L \setminus \{e, w\}$ and let K be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus e$ that contains w. If Γ_X is pruned at K, the resulting graph K' has a corresponding 2-stratifold $X_{K'}$ with nontrivial fundamental group $\pi_1(X_{K'})$ by Lemma 5.1. Now for the graph Γ_X, attach a white vertex of genus 0 with an edge of label 1 for all black vertices in P except b. There is an epimorphism from $\pi_1(X) \to \pi_1(X_{K'}) \star \mathbb{Z}_3$.

(3.) Let v be a white vertex of genus -1 and w be a white vertex of degree 3. Let L be the linear subgraph of Γ_X with terminal vertices v, w. Suppose e is the edge in L incident to w. Let P be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus e$ that contains w. Prune Γ_X at $L \cup P$. The statement follows by a similar proof to (2.) on the resulting graph Γ'.

(4.) Suppose that Γ_X contains a white vertex v of degree 2 with genus -1. We assume all other white vertices have degree ≤ 2 otherwise by the previous part X has infinite fundamental group.

Suppose that Γ_X has no black vertices of degree 3. The vertex v is not terminal and Γ_X is a linear graph. Let L_1 be the linear subgraph of Γ_X with initial vertex v and terminal vertex w where w is a terminal vertex of Γ_X. Orient the subgraph L_1 so that vertices are ordered as $w_0^1 - b_1^1 - w_1^1 - b_2^1 - ... - b_r^1 - w_r^1$ with corresponding edge labels $m_1^1 - n_1^1 - \ldots - m_r^1 - n_r^1$ where $w_0^1 = v$ and $w_r^1 = w$. Similarly, let L_2 be the linear subgraph of Γ_X with initial vertex v and terminal vertex w' where w' is the other terminal vertex of Γ_X. Orient the subgraph L_2 so that vertices are ordered as $w_0^2 - b_1^2 - w_1^2 - b_2^2 - ... - b_l^2 - w_l^2$ with corresponding edge labels $m_1^2 - n_1^2 - \ldots - m_l^2 - n_l^2$ where $w_0^2 = v$ and $w_l^2 = w'$.

Suppose that at least one L_i contains a linear subgraph T with vertices $w_j^i - b_{j+1}^i - w_{j+1}^i - b_{j+2}^i - w_{j+2}^i$ and successive labels $2, 1, 1, 2$. If T is disjoint from v then $\pi_1(X)$ surjects onto $\mathbb{Z}_2 \star \mathbb{Z}_2$. If v is a terminal vertex of T then prune Γ_X at T. Note that, there is a surjection from $\pi_1(X_\Gamma)$ to $\pi_1(X_T)$. The group $\pi_1(X_T)$ admits the following presentation:

$$\{b_1, b_2, c, \gamma : b_1^2 = 1, b_1 = b_2, b_2^2 = c, c\gamma^2 = 1\}.$$

The group $\pi_1(X_T)$ is isomorphic $\mathbb{Z}_2 \star \mathbb{Z}_2$. Therefore if the subgraph L_i of Γ_X contains a linear subgraph T then $\pi_1(X)$ is infinite.

Suppose the labeling of L_i beginning with the edge incident to v is given by $12 \ldots 12$. Prune Γ_X at the linear subgraph $w_1^1 - b_1^1 - v - b_1^2 - w_1^2$. The resulting stratifold $X_{\Gamma'}$ has vertices $w_1^1 - b_1^1 - v - b_1^2 - w_1^2$ with successive edge labels, beginning at the edge incident to w_1^1, $2, 1, 1, 2$. The 2-stratifold $X_{\Gamma'}$ has a fundamental group that admits the following presentation:

$$\{b_1, b_2, \gamma : b_1^2 = 1, b_2^2 = 1, b_1 b_2 \gamma^2 = 1\}.$$

The group $\pi_1(X_{\Gamma'})$ surjects onto $\mathbb{Z}_2 \star \mathbb{Z}_2$. Therefore for a graph Γ_X with no black vertices of degree 3 and a nonterminal white vertex of genus -1, the fundamental group of X_Γ is infinite.

Suppose that Γ contains one black vertex b of degree 3. The black vertex b is adjacent to the initial vertex w_1, w_2, w_3 of three terminal linear trees T_1, T_2, T_3 respectively. Let T_1 contain the white vertex v of genus -1 then T_2, T_3 contain only white vertices of genus 0. If either T_2, T_3 contains a subgraph $w_0 - b_1 - w_1 - b_2 - w_2$ with successive labels $2 - 1 - 1 - 2$ then $\pi_1(X_\Gamma)$ surjects onto $Z_2 * Z_2$. Otherwise, If T_2, T_3 are p-strings then apply operation $B1$ to $st(b) \cup T_2 \cup T_3$. The resulting graph Γ' is a linear 2-stratifold with a nonterminal white vertex of genus -1. Then $X_{\Gamma'}$ has infinite fundamental group and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$.

By induction, we assume that if Γ_X contains $k - 1 > 0$ black vertices of degree 3 and a nonterminal white vertex of genus -1 then $\pi_1(X_\Gamma)$ is infinite.

Assume Γ_X contains $k > 0$ black vertices of degree 3 and a nonterminal white vertex v of genus -1. Let b be a black vertex of degree 3 that is adjacent to the vertices w_1, w_2, w_3 such that w_i is the initial vertex of a terminal linear subgraph T_i for $i = 1, 2$. (The black vertex b is an outermost such vertex, in that at least two components of $\Gamma_X \setminus st(b)$ contains only vertices with degree < 3.) If v is contained in either T_1 or T_2, then by lemma 4.3.1, there exists another outermost black vertex b' of degree 3 that is adjacent to the initial vertex of two terminal linear branches that does not contain v. We assume that v is not contained in T_i. If there is a linear subgraph T with vertices $w_j - b_{j+1} - w_{j+1} - b_{j+2} - w_{j+2}$ and successive labels $2, 1, 1, 2$ contained in some T_i then there is a surjection from $\pi_1(X)$ onto $\mathbb{Z}_2 \star \mathbb{Z}_2$. If T_i are p-strings then apply operation $B1$ on $st(b) \cup T_1 \cup T_2$ such that the resulting graph Γ' has $k - 1$ black vertices of degree 3 and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$. The result follows.

(5.) Suppose that Γ_X has two white vertices w_1, w_2 of degree > 2. Let L be a linear subgraph of Γ_X with terminal vertices w_1, w_2. Let e_1 and e_2 be the edges incident to w_1 and w_2 respectively contained in L. Let P be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus \{e_1, e_2\}$ that contains neither w_1 or w_2. Allow K_i be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus e_i$ that contains w_i. If Γ_X is pruned at K_i, the resulting graph K_i' has a corresponding 2-stratifold $X_{K_i'}$ with nontrivial fundamental group $\pi_1(X_{K_i'})$ by Lemma 5.1. Now for the graph Γ_X, attach a white vertex of genus 0 with edge label one to each black vertex in the subgraph P. Then $\pi_1(X)$ surjects onto $\pi_1(X_{K_1'}) \star \pi_1(X_{K_2'})$.

□

The next corollary follows from the proof of part (4.) of the previous lemma.

Corollary 5.2.1. *If X is a pruned trivalent 2-stratifold whose graph Γ_X has a white terminal vertex of genus -1 and all edges incident to a terminal vertex have label 2 then $\pi_1(X)$ has infinite fundamental group.*

Corollary 5.2.1 is not true if we alter the condition on the terminal edge labels. For example, a trivalent linear 2-stratifold $w_0 - b_1 - w_1 - b_2 - w_3$ with successive labels $1, 2, 1, 2$, where w_0 has genus -1 and w_1, w_2 have genus 0, has fundamental group $\mathbb{Z}_8$.

Horned trees were introduced in [8]. The main property of a horned tree is that the fundamental group is $\mathbb{Z}_2$. We review the definition of a horned tree.

A **horned tree** H_T is a finite connected bipartite labelled tree such that

1. all white vertices are genus 0;
2. every black vertex b whose distance to a terminal white vertex is 1 has degree 2; otherwise b has degree 3;
3. every nonterminal white vertex has degree 2;
4. every terminal edge has label 2, every nonterminal edge has label 1;
5. there is at least one vertex of degree 3.

A trivalent linear 2-stratifold $w_0 - b_1 - w_1 - b_2 - w_3$ with successive labels $2, 1, 1, 2$, all white vertices of genus 0, and white vertex w_1 of degree 2 will also be considered a horned tree.

Lemma 5.3. *Let X be a pruned trivalent 2-stratifold where the graph Γ_X has a label 2 for all edges incident to a terminal white vertex of genus 0. Then X has infinite fundamental group if Γ_X contains one of the following:*

1. *a white vertex v of genus -1 and a horned tree H_T such that v and H_T are disjoint;*
2. *two horned trees H_1, H_2 where H_1 and H_2 are disjoint or H_1 and H_2 intersect at a vertex v such that $v = H_1 \cap H_2$ and v is a terminal vertex of H_1 and H_2;*
3. *a black terminal vertex with edge label 3 and a horned tree H_T;*
4. *a white vertex w of degree > 2 and a horned tree H_T such that either w and H_T are disjoint or w is a terminal vertex of H_T;*
5. *or a white vertex of degree > 3.*

Proof. (1.) Suppose that v and H_T are disjoint. By Lemma 5.2, v is a terminal vertex otherwise X has infinite fundamental group. Attach to each black vertex not contained in H_T a white vertex of genus 0 with edge label 1. Then there is an epimorphism from $\pi_1(X) \to \mathbb{Z}_2 \star \mathbb{Z}_2$.

(2.) Suppose that H_1 and H_2 are horned trees contained in the graph Γ_X. Attach to each black vertex not contained in H_1, H_2 of Γ_X a white vertex of genus 0 with edge label 1. Then there is an epimorphism from $\pi_1(X) \to \mathbb{Z}_2 \star \mathbb{Z}_2$.

(3.) Suppose that b is the black terminal vertex. Attach to each black vertex not contained in H_T or b a white vertex of genus 0 with edge label 1. There is an epimorphism from $\pi_1(X) \to \mathbb{Z}_2 \star \mathbb{Z}_3$.

(4.) Assume that w has degree equal to 3, all other white vertices are of degree < 3, and all white vertices have genus 0. The two main cases of this proof is when H_T is disjoint from w and when w is a terminal vertex of H_T.

Suppose that H_T is disjoint from w. Let L be the linear subgraph of Γ_X with terminal vertices w and v where v is a terminal vertex of H_T such that $L \cap H_T = v$. Let e_1, e_2 be the edges incident to w, v (respectively) that are contained in L. Allow the subgraph P to be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus \{e_1, e_2\}$ that contains $L \setminus \{e_1, e_2, w, v\}$. Also allow the subgraph R to be the subgraph of Γ_X that corresponds to the component of $\Gamma_X \setminus \{e_1\}$ that contains w. If Γ_X is pruned at R, the resulting graph R' has a corresponding 2-stratifold $X_{R'}$ with nontrivial fundamental group $\pi_1(X_{R'})$ by lemma 5.1. Prune Γ_X at $R \cup e_1 \cup e_2 \cup P \cup H_T$ and attach white vertices of genus 0 with edge label 1 to all black vertices contained in P of the pruned graph. The resulting graph Γ' has a fundamental group isomorphic to $\pi_1(X_{R'}) \star \pi_1(\mathbb{Z}_2)$.

Now suppose that w is a terminal vertex of H_T and let e_1, e_2 be the edges incident to w that are not contained in H_T. Allow the subgraph of Γ_X corresponding to the component of $\Gamma_X \setminus e_i$ that does not contain H_T be called D_i. Let $E_i = D_i \cup e_i \cup w$. By part (2.), if E_i contains a horned tree then $\pi_1(X)$ is infinite, so we assume that E_i contains no horned trees. Prune Γ_X at $E_1 \cup E_2 \cup H_T$ and let the resulting graph be called Γ'. We now show that the fundamental group of $X_{\Gamma'}$ is infinite. Therefore the fundamental group of X_Γ will be infinite.

If Γ' contains no black vertices of degree 3 then Γ' has a single white vertex w of degree 3 where w is a terminal vertex of H_T and w is the initial vertex of two terminal p-strings E_1, E_2 of length $2p, 2q$. The associated 2-stratifold $X_{\Gamma'}$ has fundamental group that can be represented with the following presentation:

$$\{c_1, c_2, c_3 : c_1^{2^p} = 1, c_2^{2^q} = 1, c_3^2 = 1, c_1c_2c_3^2 = 1\}.$$

The fundamental group $\pi_1(X_{\Gamma'})$ surjects onto $\mathbb{Z}_2 \star \mathbb{Z}_2$. Therefore if Γ' contains no black vertices of degree 3 then the fundamental group of $X_{\Gamma'}$ is infinite.

We proceed by induction. Assume that if Γ' contains $k-1>0$ black vertices of degree 3 then $\pi_1(X_{\Gamma'})$ is infinite.

Suppose that Γ' has $k>0$ black vertices of degree 3. Let b be a black vertex of degree 3 that is adjacent to the vertices w_1, w_2, w_3 such that v_i is the initial vertex of a terminal linear subgraph T_i for $i = 1, 2$. (The black vertex b is an outermost such vertex, in that at least two components of $\Gamma_X \setminus st(b)$ contains only vertices with degree < 3.) If the terminal linear graphs T_i are contained in E_i or H_T then they are p-strings. Apply operation $B1$ on $st(b) \cup T_1 \cup T_2$ such that the resulting graph Γ'' has $k-1$ black vertices of degree 3 and $\pi_1(X_{\Gamma}') \cong \pi_1(X_{\Gamma''})$. The result follows.

(5.) Suppose that w is the white vertex of degree 4 contained in Γ_X. Then Γ_X contains all white terminal vertices and all white vertices of genus 0, otherwise X has infinite fundamental group.

Suppose that Γ_X has no black vertices of degree 3. Let e_i be the edges incident to w for $1 \leq i \leq 4$. Define L_i to be the linear subgraph whose intial vertex is w, whose terminal vertex is a terminal vertex of Γ_X, and L_i contains the edge e_i. If at least one L_i contains a horned tree then X_Γ has infinite fundamental group. Assume then that each L_i is a p-string of length $2p_i$. The 2-stratifold X_Γ has fundamental group that can be represented with the following presentation:

$$\{c_1, c_2, c_3, c_4 : c_1^{2^{p_1}} = 1, c_2^{2^{p_2}} = 1, c_3^{2^{p_3}} = 1, c_4^{2^{p_4}} = 1, c_1c_2c_3c_4 = 1\}.$$

This is an infinite F-group.

Suppose that Γ_X has $k>0$ black vertices of degree 3. Let b be a black vertex of degree 3 that is adjacent to the vertices w_1, w_2, w_3 such that v_i is the initial vertex of a terminal linear subgraph T_i for $i = 1, 2$. (The black vertex b is an outermost such vertex, in that at least two components of $\Gamma_X \setminus st(b)$ contains only vertices with degree < 3.) If T_i contains a horned tree then X_Γ has infinite fundamental group. We assume that the terminal linear subgraphs T_i are p-strings. Apply operation $B1$ on $st(b) \cup T_1 \cup T_2$ such that the resulting graph Γ' has $k-1$ black vertices of degree 3 and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$. The result follows by the induction hypothesis.

□

Theorem 5.4. *Let X be a pruned trivalent 2-stratifold where the graph Γ_X has a label 2 for all edges incident to a terminal white vertex of genus 0. If X has finite fundamental group then Γ_X is a tree that satisfies one of the following conditions:*

1. *Γ_X has one terminal vertex v of genus -1 whose incident edge label is 1 while all other white vertices are genus 0, all terminal vertices are white, all white vertices are of degree ≤ 2, and Γ_X contains no horned trees;*
2. *Γ_X has all white vertices of genus 0, all terminal vertices are white, and there is exactly one white vertex v of degree 3 while all other white vertices are of degree < 3, and Γ_X contains no horned tree H_T such that either v and H_T are disjoint or v is a terminal vertex of H_T;*
3. *Γ_X has all white vertices are genus 0, all terminal vertices are white, all white vertices are of degree ≤ 2, and Γ_X contains at most one horned tree;*
4. *Γ_X has all white vertices are genus 0, one black terminal vertex, all white vertices are of degree ≤ 2, and Γ_X contains no horned tree.*

Proof. If Γ_X contains exactly one white vertex v of genus -1 then v is terminal by lemma 5.2 and the label incident to v is 1 by corollary 5.2.1. Further, all white vertices of Γ_X are of degree < 3 by lemma 5.2 and Γ_X contains no horned trees by Lemma 5.3.

If Γ_X contains all white vertices of genus 0 and all terminal vertices are white then there exists at most one white vertex v of degree > 2 by lemma 5.2. If all white vertices of Γ_X are of degree < 3 then Γ_X contains at most one horned tree by lemma 5.3. If Γ_X contains a white vertex v of degree > 2 then v is degree 3 and Γ_X contains no horned tree H_T such that either v and H_T are disjoint or v is a terminal vertex of H_T by Lemma 5.3.

If Γ_X contains exactly one black terminal vertex then Γ_X must have all white vertices of degree < 3 by Lemma 5.2 and Γ_X cannot contain a horned tree H_T by Lemma 5.3.

□

6 Labellings of Trivalent 2-Stratifolds with Finite Fundamental Group

We find sufficient conditions for a trivalent 2-stratifold X to have finite fundamental group. These conditions are given by the following: lemma 6.2; lemma 6.3; lemma 6.4; lemma 6.5. To find the sufficient conditions, we will inductively apply operation $B1$ to a graph Γ_X that satisfies a set of conditions from theorem 5.4.

The figure below is an example of a graph Γ that satisfies a set of conditions given by Theorem 5.4. The fundamental group of X_Γ is $\mathbb{Z}_{16}$. The order of this fundamental group is determined by the linear subgraph with initial vertex given by the genus -1 vertex and terminal vertex given by t_1. The connected subgraphs of Γ that are composed of red edges along with incident vertices are terminal p-strings. We use this example as motivation for the definition of an O-string.

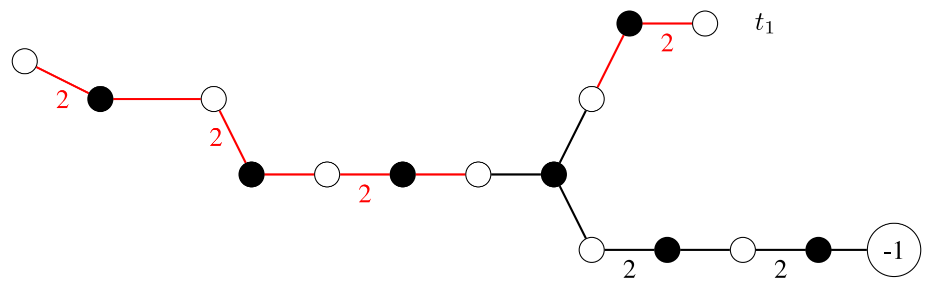

Figure 3: The graph Γ.

An O**-string** of length $2r$ is an oriented linear graph $w_0 - b_1 - w_1 - b_2 - ... - b_r - w_r$ where the genus of w_0 is either 0 or -1 while all other white vertices w_i are of genus 0, the labels m_i, n_i for the successive edges of $w_{i-1} - b_i - w_i$ are either $m_i = 1, n_i = 1$ or $m_i = 1, n_i = 2$ for $0 < i < r$, and the labels m_r, n_r for the edges of $w_{r-1} - b_r - w_r$ are given by the labels $m_r = 1, n_r = 2$. A terminal p-string is an O-string.

The next lemma observes certain subgraphs of a given O-string are preserved under operation $B1$. For example, the graph Γ' below is obtained by applying operation $B1$ to the graph Γ in the above figure. The linear subgraph with initial vertex given by the genus -1 vertex and terminal vertex given by t_1 is an O-string in both Γ and Γ' and contains the same number of edges with label 2. The subgraph composed of red edges and incident vertices in Γ' is the terminal associated p-string in Γ'.

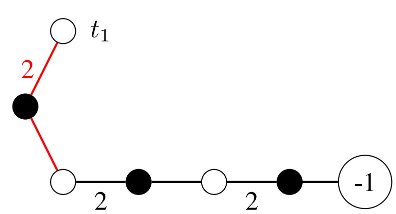

Figure 4: The graph Γ' obtained from applying operation $B1$ to Γ.

Lemma 6.1. *Let X be a trivalent 2-stratifold whose graph Γ_X is a tree that contains $n \geq 1$ black vertices of degree 3. Let b be a black vertex of degree 3 with adjacent vertices v_1, v_2, v_3, such that v_i is the initial vertex of a terminal p-string P_i of length $2p_i$ for $i = 1, 2$. Let Γ' be obtained from Γ by operation $B1$ at $st(b) \cup P_1 \cup P_2$. Let P' be the associated p-string in Γ'.*

Let L_i be a linear subgraph of Γ_X with an initial vertex w which is a white vertex not contained in P_i and a terminal vertex t_i where t_i is the terminal vertex of P_i and a terminal vertex of Γ_X. Let L' be a linear subgraph of Γ' with initial vertex w not contained in $P' \setminus w_3$ and terminal vertex t' where t' is the terminal vertex of P' and a terminal vertex of Γ'.

1. *If L' is an O-string then L_1, L_2 are O-strings.*
2. *If L' is an O-string that contains k edges with label 2 then L_1, L_2 contains $r \geq k$ edges with label 2 and at least one L_i has k edges with label 2.*
3. *If Γ' contains a horned tree $H_{T'}$ then Γ_X contains a horned tree H_T.*
4. *If a horned tree $H_{T'}$ of Γ' contains a terminal vertex of Γ' then a horned tree H_T of Γ_X contains a terminal vertex of Γ_X.*

Proof. (1.) Suppose L' is an O-string. Let S be the linear subgraph $w_0 - b_1 - w_1 - b_2 - ... - b_r - w_r$ of L' with initial vertex $w_0 = w$ and terminal vertex $w_r = v_3$. For $1 \leq i \leq r$, the labels m_i, n_i for the successive edges of $w_{i-1} - b_i - w_i$ contained in S are either $m_i = 1, n_i = 1$ or $m_i = 1, n_i = 2$. Let N_i be the linear subgraph of L_i with initial vertex v_3 and terminal vertex t_i. The subgraph N_i is an O-string. The subgraph L_i is composed of the subgraph S with initial vertex w and terminal vertex v_3 followed by the subgraph N_i with initial vertex v_3 and terminal vertex t_i. The linear graph L_i is an O-string.

(2.) Suppose that L' is an O-string that contains k edges with label 2. Let S and N_i be linear subgraphs as definied in (1.). By the previous proof L_i is an O-string. The subgraph S has $r' \geq 0$ edges with label 2. The subgraph P' has k' edges with label 2 where $k' + r' = k$. The integer k' is the minimum of $\{p_1, p_2\}$. Therefore for some i, N_i has k' edges with label 2. Then the linear graph L_i has $k' + r' = k$ edges with label 2.

(3.) Suppose Γ' contains a horned tree $H_{T'}$. For the terminal p-string P' of Γ', order the vertices $w'_0 - b'_1 - w'_1 - b'_2 - ... - b'_r - w'_r$ so that the initial vertex w'_0 is v_3 and w'_r is the terminal vertex t' of Γ'. The horned tree $H_{T'}$ is disjoint from P' or intersects P'. If the horned tree $H_{T'}$ is disjoint from P' then $H_{T'}$ is contained in Γ_X.

Suppose that $H_{T'}$ intersects P'. Then $H_{T'}$ intersects P' at only the vertex v_3 or along the linear subgraph P'' with initial vertex v_3 and terminal vertex w'_1. The linear subgraph P'' has vertices $w'_0 - b'_1 - w'_1$ where $w'_0 = v_3$ and successive labels $1, 2$. If the horned tree $H_{T'}$ intersects the subgraph of P' only at v_3 then $H_{T'}$ is contained in Γ_X. Suppose that the horned tree $H_{T'}$ contains the subgraph P'' of P'. Let H be a subgraph of $H_{T'}$ where $H = H_{T'} \setminus (st(b'_1) \cup w'_1)$. Then H is contained in Γ_X. For the terminal p-strings P_i of Γ_X, order the vertices $w^i_0 - b^i_1 - w^i_1 - b^i_2 - ... - b^i_{r_i} - w^i_{r_i}$ where $w^i_0 = v_i$ and $w^i_{r_i} = t_i$ of Γ_X for $i = 1, 2$ and define E_i to the linear subgraph of Γ_X with initial vertex v_3 and terminal vertex w^i_1. Then $H \cup E_1 \cup E_2$ is a horned tree contained in Γ_X.

(4.) Suppose Γ' contains a horned tree $H_{T'}$ where $H_{T'}$ contains a terminal vertex of Γ'. Let w be a terminal vertex of Γ' that is contained in $H_{T'}$. If P' is disjoint from $H_{T'}$ then $H_{T'}$ is contained in Γ_X and w is a terminal vertex of Γ_X and $H_{T'}$. We assume that P' is not disjoint from $H_{T'}$.

Suppose that w is disjoint from P'. Let H be the subgraph of $H_{T'}$ as defined in part (3.). The vertex w is contained in H and either $H_{T'}$ is contained in Γ_X or the horned tree $H_T = H \cup E_1 \cup E_2$ is contained in Γ_X where H, E_i are defined as in part (3.). If $H_{T'}$ is contained in Γ_X then w is a terminal vertex of Γ_X and $H_{T'}$. If H_T is contained in Γ_X then w is a terminal vertex of Γ_X and H_T.

Suppose that w is contained in P'. Then P' is a p-string of length 2 with initial vertex v_3 and terminal vertex w. It follows from (2.) that at least one of the terminal linear branches P_i in Γ_X is p-string of length 2. The horned tree $H \cup E_1 \cup E_2$ contains a terminal vertex of Γ_X.

□

The proofs for lemma 6.2 until lemma 6.5 are similar with only minor alterations. Each proof contains three cases for a trivalent 2-stratifold X: either Γ_X contains no black vertices of degree 3; Γ_X contains one black vertex of degree 3; or Γ_X contains $k > 1$ black vertices of degree 3. We will show all cases for Lemma 6.2. Then for Lemma 6.3 until lemma 6.5, we will abbreviate the proofs by showing the cases when Γ_X contains no black vertices of degree 3 or $k > 1$ black vertices of degree 3.

Lemma 6.2. *Let X be a pruned trivalent 2-stratifold where Γ_X has a label 2 for all edges incident to a terminal white vertex of genus 0. Let Γ_X have all white vertices of genus 0, all terminal vertices are white, and all white vertices are of degree ≤ 2. If $\pi_1(X)$ is finite then all of the following hold:*

1. *Γ_X contains a horned tree H_T.*
2. *If L is a linear subgraph of Γ_X whose initial vertex v is a terminal vertex of H_T and whose terminal vertex w is a terminal vertex of Γ_X where $L \cap H_T = v$ and $w \neq v$ then L is an O-string.*

3. *The fundamental group $\pi_1(X)$ is isomorphic to $\mathbb{Z}_{2^{k+1}}$ where the integer $k = 0$ if H_T contains a terminal vertex of Γ_X and $k > 0$ otherwise. The integer $k > 0$ corresponds to the minimal number of edges with label 2 in all linear subgraphs L whose initial vertex v is a terminal vertex of H_T and whose terminal vertex w is a terminal vertex of Γ_X where $L \cap H_T = v$ and $w \neq v$.*

Proof. It follows by theorem 5.4, the fundamental group $\pi_1(X)$ is finite implies that the graph Γ_X is a tree that contains at most one horned tree.

Suppose that Γ_X has no black vertices of degree 3. The graph Γ_X is a linear graph. Orient the graph Γ_X so that vertices are ordered as $w_0 - b_1 - w_1 - b_2 - ... - b_r - w_r$ with corresponding edge labels $m_1 - n_1 - \ldots - m_r - n_r$. By assumption the subgraph $w_0 - b_1 - w_1$ has successive labels $m_1 = 2, n_1 = 1$ and the subgraph $w_{r-1} - b_r - w_r$ has successive labels $m_r = 1, n_r = 2$. Each subgraph $w_{i-1} - b_i - w_i$ for $1 < i < r$ has successive labels $m_i = 2, n_i = 1$ or $m_i = 1, n_i = 2$. There exists a j, where $1 < j \leq r$, such that $w_{j-2} - b_{j-1} - w_{j-1}$ has successive labels $m_{j-1} = 2, n_{j-1} = 1$ and $w_{j-1} - b_j - w_j$ has successive labels $m_j = 1, n_j = 2$. The graph Γ_X contains a horned tree H given by the graph $w_{j-2} - b_{j-1} - w_{j-1} - b_j - w_j$. By lemma 5.3, Γ_X does not contain any other horned tree.

Suppose H does not contain a vertex that is terminal in Γ_X. Let L_1 be the linear subgraph of Γ_X with initial vertex w_{j-2} and terminal vertex w_0 and let L_2 be the linear subgraph of Γ_X with initial vertex w_j and terminal vertex w_r. The linear subgraphs L_1, L_2 are p-strings of length $2p_1, 2p_2$. Otherwise Γ_X contains more than one horned tree. Note that L_1, L_2 are O-strings. L-prune Γ_X at the linear subgraphs L_1 and L_2. The resulting graph Γ' is a linear graph where $\Gamma' = \Gamma'(2^{p_1}, 1, 2, 1, 1, 2, 1, 2^{p_2})$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma'})$. A presentation of the fundamental group of $X_{\Gamma'}$ is given by:

$$\{x_1, x_2, x_3, x_4 : x_1^{2^{p_1}} = 1, x_1 = x_2^2, x_2 = x_3, {x_3}^2 = x_4, x_4^{2^{p_2}} = 1\}.$$

This presentation is equivalent to:

$$\{x_3 : x_3^{2^{p_1+1}} = 1, x_3^{2^{p_2+1}} = 1\}.$$

This group is finite cyclic of order given by the min$(2^{p_1+1}, 2^{p_2+1})$. Therefore $\pi_1(X) \cong \mathbb{Z}_{2^{k+1}}$ where k is the minimum of $\{p_1, p_2\}$. The number k is the minimum number of edges with label 2 in the O-strings L_1, L_2.

Suppose that H contains a vertex that is terminal in Γ_X. Assume that the horned graph H is $w_0 - b_1 - w_1 - b_2 - w_2$. The linear subgraph L of Γ_X with initial vertex w_2 and terminal vertex w_r is p-string of order $2(r-2) = 2p_1$ (and hence an O-string). L-prune Γ_X at the linear graph L. The resulting graph Γ' is a linear graph (with terminal white vertices) where $\Gamma' = \Gamma'(2, 1, 1, 2, 1, 2^{p_1})$. A presentation of the fundamental group of $X_{\Gamma'}$ is given by:

$$\{x_1, x_2, x_3 : x_1^2 = 1, x_1 = x_2, x_2^2 = x_3, x_3^{2^{p_1}} = 1\}.$$

This presentation is equivalent to:

$$\{x_1 : x_1^2 = 1\}.$$

Therefore $\pi_1(X) \cong \mathbb{Z}_2$ if H contains a terminal vertex of Γ_X.

We now show that this lemma holds for a graph Γ_X with one black vertex of degree 3 then proceed with induction for a graph Γ_X with $n > 1$ black vertices of degree 3.

Suppose that Γ_X contains one black vertex b of degree 3. The black vertex b is adjacent to the initial vertex v_1, v_2, v_3 of three terminal linear subgraphs T_1, T_2, T_3 respectively. At most one terminal linear subgraph T_1, T_2, T_3 contains a horned tree. If T_i does not contain a horned tree then T_i is a p-string. Let T_1, T_2 be p-strings. Let the terminal vertices of T_i which are terminal vertices of Γ_X be called t_i for $i = 1, 2$. Apply operation $B1$ to $st(b) \cup T_1 \cup T_2$. The resulting graph Γ' is a linear 2-stratifold. Let the associated p-string be called T'. Note that v_3 is the initial vertex of the associated p-string T' in Γ' and v_3 is not a terminal vertex of either Γ_X or Γ'. The fundamental group $\pi_1(X_{\Gamma'})$ is isomorphic to $\mathbb{Z}_{2^{k+1}}$ for $k \geq 0$ and Γ' contains a horned tree H'. Orient the graph Γ' so that vertices are ordered as $w'_0 - b'_1 - w'_1 - b'_2 - ... - b'_r - w'_r$ with corresponding edge labels $m'_1 - n'_1 - \ldots - m'_r - m'_r$. Then there is a j, where $1 < j \leq r$ such that $w'_{j-2} - b'_{j-1} - w'_{j-1} - b'_j - w'_j$ is a horned tree H'.

The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$ and by Lemma 6.1 if Γ' contains a horned tree H' then Γ_X contains a horned tree H. Further if $\pi_1(X_{\Gamma'})$ is isomorphic to $\mathbb{Z}_2$ then the horned tree H' of Γ' contains a terminal vertex of Γ'. It follows that $\pi_1(X_\Gamma)$ is isomorphic to $\mathbb{Z}_2$ and by Lemma 6.1 the horned tree H contains a terminal vertex of Γ_X.

We now show that all linear subgraphs L of Γ_X whose initial vertex v is a terminal vertex of H and whose terminal vertex w is a terminal vertex of Γ_X where $H \cap L = w$ and $v \neq w$ are O-strings. Then we show that if $\pi_1(\Gamma_X) \cong \mathbb{Z}_{2^{k+1}}$ where $k > 0$ that k corresponds to the minimal number of edges with label 2 in all O-strings L with initial vertex v and terminal vertex w.

Suppose that $\pi_1(X_{\Gamma'}) \cong \mathbb{Z}_2$. Let the horned tree H' be the subgraph $w'_0 - b'_1 - w'_1 - b'_2 - w'_2$ in Γ'. Let L' be the linear subgraph of Γ' with initial vertex w'_2 and terminal vertex w'_r. The vertex v_3 is either a nonterminal vertex of H', a terminal vertex of H', or disjoint from H'.

If v_3 is disjoint from H' in Γ' then $v_3 = w'_i$ where $2 < i < r$ and H' is properly contained in the terminal linear subgraph T_3 of Γ_X. If v_3 is a terminal vertex of H' then $v_3 = w'_2$ and the horned tree H' is the terminal linear subgraph T_3 of Γ_X. Since the linear subgraph L' is a p-string in Γ', it follows by Lemma 6.1, that every linear subgraph L of Γ_X whose initial vertex is w'_2 and whose terminal vertex is t_i of Γ_X is an O-string.

If v_3 is a nonterminal vertex of H' then $v_3 = w'_1$. The horned tree H contained in Γ_X contains the black vertex b. Therefore the terminal linear branches T_1, T_2, T_3 are all p-strings. T_3 is of length 2. If T_i is of length > 2 then let O_i be the linear subgraph contained in T_i whose initial vertex v is a terminal vertex of H and whose terminal vertex is a terminal vertex of Γ_X such that $O_i \cap H = v$. Then O_i is a p-string.

Suppose that $\pi_1(X_{\Gamma'}) \cong \mathbb{Z}_{2^{k+1}}$ where $k > 0$. Then H' is the subgraph of Γ' with vertices $w'_{j-2} - b'_{j-1} - w'_{j-1} - b'_j - w'_j$ where $2 < j < r$. The horned tree H' does not contain a terminal vertex of Γ'. Let L'_1 be the linear subgraph of Γ' with initial vertex w'_{j-2} and terminal vertex w'_0 and let L'_2 be the linear subgraph of Γ' with initial vertex w'_j and terminal vertex w'_r. The linear subgraphs L'_1, L'_2 are p-strings of length $2p'_1, 2p'_2$ where $p'_i \geq k$ and for at least one L'_i we have $p'_i = k$. Suppose that v_3 is contained in the linear graph whose initial vertex is w'_{j-1} and whose terminal vertex is w'_r. (If v_3 is contained in the linear graph whose initial vertex is w'_{j-1} and whose terminal vertex is w'_0 then the same argument applies.) The vertex v_3 is either a nonterminal vertex of H', a terminal vertex of H', or disjoint from H'.

If v_3 is disjoint from H' in Γ' then $v_3 = w'_i$ where $j < i < r$ and H' is properly contained in the terminal linear subgraph T_3 of Γ_X. If v_3 is a terminal vertex of H' then $v_3 = w'_j$ and H' is properly contained in the terminal linear subgraph T_3 of Γ_X. In both cases since the linear subgraph L'_2 in Γ' is a p-string, it follows by Lemma 6.1, that every linear subgraph L of Γ_X whose initial vertex is w'_j and whose terminal vertex t_i of Γ_X is an O-string. L'_1 is a p-string in Γ' that is disjoint from T'. By remark 4.2, L'_1 is contained in Γ_X. Let R_i be a linear subgraph of Γ_X whose initial vertex is w'_j and whose terminal vertex is t_i. If L'_2 contains k edges with label 2 then at least one R_i for $i = 1, 2$ contains k edges with label 2. If L'_2 does not contain k edges with label 2 then R_i contains more than k edges with label 2. Then the subgraph L'_1 of Γ' contains k edges with label 2. By remark 4.2, L'_1 is contained in Γ_X.

If v_3 is a nonterminal vertex of H' then $v_3 = w'_{j-1}$. The horned tree H contained in Γ_X contains the black vertex b. Therefore the terminal linear branches T_1, T_2, T_3 are all p-strings. By the same argument in the previous paragraph, all terminal p-strings T_i are of length l where $l \geq 2(k+1)$ and at least one T_i is of length $2(k+1)$.

The lemma holds for a graph Γ_X with one black vertex of degree 3. We now proceed with induction for a graph Γ_X with $n > 1$ black vertices of degree 3.

Suppose that Γ_X contains $n > 1$ black vertices of degree 3. Let b be a black vertex of degree 3 that is adjacent to the vertices v_1, v_2, v_3 such that v_i is the initial vertex of a terminal linear subgraph T_i for $i = 1, 2$. (The black vertex b is an outermost such vertex, in that at least two components of $\Gamma_X \setminus st(b)$ contains only vertices with degree < 3.) If T_i does not contain a horned tree then T_i is a p-string. If either T_1 or T_2 contains a horned tree, then by lemma 4.3.1, there exists another such black vertex b' of degree 3 that is adjacent to the initial vertices of two terminal linear branches T'_1, T'_2. Since X_Γ has finite fundamental group the two terminal linear branches T'_1, T'_2 do not contain a horned tree. We assume T_1 and T_2 do not contain a horned tree. Then T_1 and T_2 are terminal p-strings. Let the terminal vertices of T_i which are terminal vertices of Γ_X be called t_i for $i = 1, 2$. Apply operation $B1$ to $st(b) \cup T_1 \cup T_2$. The resulting graph Γ' has $n - 1$ black vertices of degree 3. Let the associated p-string be called T' and let the terminal vertex of T' and Γ' be called t'. By the induction hypothesis, $\pi_1(X_{\Gamma'})$ is isomorphic to $\mathbb{Z}_{2^{k+1}}$ for $k \geq 0$ and Γ' contains a horned tree H'.

The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$ and by Lemma 6.1 if Γ' contains a horned tree H' then Γ_X contains a horned tree H. Further if $\pi_1(X_{\Gamma'})$ is isomorphic to $\mathbb{Z}_2$ then the horned tree H' of Γ' contains a terminal vertex of Γ'. By Lemma 6.1, this implies that $\pi_1(X_\Gamma)$ is isomorphic to $\mathbb{Z}_2$ and the horned tree H contains a terminal vertex of Γ_X.

Let L' be a linear subgraph of Γ' whose initial vertex v' is a terminal vertex of H' and whose terminal vertex w' is a terminal vertex of Γ' where $L' \cap H' = v'$ and $w' \neq v'$. By the induction hypothesis L' is an O-string. By remark 4.2 and lemma 6.1, if L' is disjoint from $T' \setminus v_3$ then L' is an O-string in Γ_X that is disjoint from $st(b) \cup T_1 \cup T_2$ and the initial vertex v' of L' is a terminal vertex of H_T. We assume L' is not disjoint from $T' \setminus v_3$. Then the terminal vertex

w' of L' is t' which is the terminal vertex of T'. The vertex v_3 is either a nonterminal vertex of H', a terminal vertex of H', or disjoint from H'.

If v_3 is disjoint from H' then L' properly contains the p-string T'. If v_3 is a terminal vertex of H' then L' is the p-string T'. In both cases H' is contained in Γ_X. It follows by Lemma 6.1, that every linear subgraph L of Γ_X whose initial vertex is v' and whose terminal vertex is t_i of Γ_X is an O-string.

If v_3 is a nonterminal vertex of H' then L' is properly contained in T'. For the terminal p-strings T_i of Γ_X, order the vertices $w_0^i - b_1^i - w_1^i - b_2^i - ... - b_{r_i}^i - w_{r_i}^i$ where $w_0^i = v_i$ and $w_{r_i}^i = t_i$ of Γ_X for $i = 1, 2$. The terminal linear subgraphs T_1, T_2 of Γ_X intersect the horned tree H at the subgraphs $w_0^i - b_1^i - w_1^i$. The terminal linear subgraphs of T_1, T_2 whose initial vertex is w_1^i and whose terminal vertex is t_i is an O-string.

Suppose that $\pi_1(X_\Gamma) \cong \mathbb{Z}_{2^{k+1}}$ where $k > 0$. Then H' does not contain a terminal vertex of Γ'. By the induction hypothesis, there exists an O-string L' of Γ' whose initial vertex is a terminal vertex v' of H' and whose terminal vertex w' is a terminal vertex of Γ' where $L' \cap H' = v'$ and L' contains k edges with label 2. The number k is minimal among all such O-strings. By remark 4.2 and lemma 6.1, if L' is disjoint from $T' \setminus v_3$ then L' is an O-string in Γ_X that is disjoint from $st(b) \cup T_1 \cup T_2$ and the initial vertex v' of L' is a terminal vertex of H. We assume L' is not disjoint from $T' \setminus v_3$.

The vertex v_3 is either a nonterminal vertex of H', a terminal vertex of H', or disjoint from H'. If v_3 is disjoint from H' then L' properly contains the p-string T'. If v_3 is a terminal vertex of H' then L' is the p-string T'. In both cases H' is contained in Γ_X. It follows by Lemma 6.1, that at least one linear subgraph L of Γ_X whose initial vertex is v' and whose terminal vertex is t_i of Γ_X is an O-string with k edges with label 2.

If v_3 is a nonterminal vertex of H' then L' is properly contained in T'. The terminal linear subgraph T' contains $k+1$ edges with label 2. For the terminal p-strings T_i of Γ_X, order the vertices $w_0^i - b_1^i - w_1^i - b_2^i - ... - b_{r_i}^i - w_{r_i}^i$ where $w_0^i = v_i$ and $w_{r_i}^i = t_i$ of Γ_X for $i = 1, 2$. The terminal linear subgraphs T_1, T_2 of Γ_X intersect the horned tree H at the subgraphs $w_0^i - b_1^i - w_1^i$ and by lemma 6.1 at least one of the terminal linear subgraph T_1, T_2 contains $k+1$ edges with label 2. Therefore at least one of the terminal linear subgraphs of T_1, T_2 whose initial vertex is w_1^i and whose terminal vertex is t_i is an O-string with k edges with label 2.

□

Lemma 6.3. *Let X be a pruned trivalent 2-stratifold where Γ_X has a label 2 for all edges incident to a terminal white vertex of genus 0. Let Γ_X have one white terminal vertex of genus -1 with incident edge label 1 while all other white vertices are genus 0, all terminal vertices are white, and all white vertices are of degree ≤ 2. If $\pi_1(X)$ is finite then all of the following hold:*

1. *Let L be a linear subgraph of Γ_X whose initial vertex v is the white vertex of genus -1 and whose terminal vertex w is a terminal vertex of Γ_X where $w \neq v$. Then L is an O-string.*
2. *The fundamental group $\pi_1(X)$ is isomorphic to $\mathbb{Z}_{2^{k+1}}$ where the integer $k > 0$ corresponds to the minimal number of edges with label 2 in all L whose initial vertex v is the white vertex of genus -1 and whose terminal vertex w is a terminal vertex of Γ_X where $w \neq v$.*

Proof. It follows by theorem 5.4, the fundamental group $\pi_1(X)$ is finite implies Γ_X is a tree that contains no horned trees. Let v be the terminal white vertex of genus -1.

Suppose that Γ_X has no black vertices of degree 3. The graph Γ_X is a linear graph. Orient the graph Γ_X so that vertices are ordered as $w_0 - b_1 - w_1 - b_2 - ... - b_r - w_r$ with corresponding edge labels $m_1 - n_1 - \ldots - m_r - n_r$ and $w_0 = v$. By assumption the labels $m_1 = 1, n_1 = 2$ and $m_r = 1, n_r = 2$. If there exists a subgraph $w_{i-1} - b_i - w_i$ for $1 < i < r$ with successive labels $m_i = 2, n_i = 1$ then Γ_X contains a horned tree. Therefore each subgraph $w_{i-1} - b_i - w_i$ for $1 < i < r$ has successive labels $m_i = 1, n_i = 2$. The graph Γ_X is an O-string. L-prune Γ_X, the resulting graph Γ' is a linear graph with vertices $w_0 - b'_1 - w'_1$ where $\Gamma' = \Gamma'(1, 2^r)$ and w_0 has genus -1. A presentation of the fundamental group of $X_{\Gamma'}$ is given by:

$$\{x_1, y, c : x_1^{2^r} = 1, x_1 = c, cy^2 = 1\}.$$

This presentation is equivalent to:

$$\{y : y^{2^{r+1}} = 1\}.$$

Then $\pi_1(X) \cong \mathbb{Z}_{2^{r+1}}$ where r is the number of edges with label 2 in the O-string Γ_X.

Suppose that Γ_X contains $n > 1$ black vertices of degree 3. Let b be a black vertex of degree 3 that is adjacent to the vertices v_1, v_2, v_3 such that v_i is the initial vertex of a terminal linear subgraph T_i for $i = 1, 2$. (The black vertex b is an outermost such vertex, in that at least two components of $\Gamma_X \setminus st(b)$ contains only vertices with degree < 3.) If T_i does not contain v then T_i is p-string. If T_i contains v then by lemma 4.3.1, there exists another outermost black vertex b' of degree 3 that is adjacent to the initial vertex of two terminal linear branches T_1', T_2'. Then T_1' and T_2' are terminal p-strings. We assume that both T_1 and T_2 are terminal p-strings. Let the terminal vertices of T_i which are terminal vertices of Γ_X be called t_i for $i = 1, 2$. Apply operation $B1$ to $st(b) \cup T_1 \cup T_2$. The resulting graph Γ' has $n - 1$ black vertices of degree 3. Let the associated p-string be called T' and let the terminal vertex of T' and Γ' be called t'.

By the induction hypothesis, $\pi_1(X_{\Gamma'})$ is isomorphic to $\mathbb{Z}_{2^{k+1}}$ for $k > 0$. The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$.

Let L' be a linear subgraph of Γ' whose initial vertex is v and whose terminal vertex w' is a terminal vertex of Γ' where $v \neq w'$. By the induction hypothesis L' is an O-string. If L' is disjoint from $T' \setminus v_3$ then L' is disjoint from T'. By remark 4.2, L' is an O-string in Γ_X that is disjoint from $v_3 \cup st(b) \cup T_1 \cup T_2$. We assume L' is not disjoint from $T' \setminus v_3$. Then the terminal vertex w' of L' is t' which is the terminal vertex of T'. By Lemma 6.1 it follows that every linear subgraph L of Γ_X whose initial vertex is v and whose terminal vertex is t_i of Γ_X is an O-string.

By the induction hypothesis, there exists an O-string L' of Γ' whose initial vertex is v and whose terminal vertex w' is a terminal vertex of Γ' where $v \neq w'$ and L' contains $k > 0$ edges with label 2. The number k is minimal among all such O-strings. If L' is disjoint from $T' \setminus v_3$ then L' is disjoint from T'. By remark 4.2, L' is an O-string in Γ_X that is disjoint from $v_3 \cup st(b) \cup T_1 \cup T_2$. We assume L' is not disjoint from $T' \setminus v_3$. Then the terminal vertex w' of L' is t' which is the terminal vertex of T'. By Lemma 6.1 there exists an O-string of Γ_X whose initial vertex is v and whose terminal vertex is t_i of Γ_X with exactly k edges with label 2 for some $i = 1, 2$.

□

The proofs for lemma 6.4 lemma and 6.3 are similar for the case when Γ_X contains $k > 1$ black vertices of degree 3. We will abbreviate the proof for lemma 6.4 by only showing the case when Γ_X contains no black vertices of degree 3

Lemma 6.4. *Let X be a pruned trivalent 2-stratifold where Γ_X has a label 2 for all edges incident to a terminal white vertex of genus 0. Let Γ_X have all white vertices of genus 0, one black terminal vertex, and all white vertices are of degree ≤ 2. If $\pi_1(X)$ is finite then all of the following hold:*

1. *Let L be a linear subgraph of Γ_X whose initial vertex v is the white vertex adjacent to the black terminal vertex and whose terminal vertex w is a white terminal vertex of Γ_X. Then L is an O-string.*
2. *The fundamental group $\pi_1(X)$ is isomorphic to $\mathbb{Z}_{3(2^k)}$ where the integer $k > 0$ corresponds to the minimal number of edges with label 2 in all L whose initial vertex v is the white vertex adjacent to the black terminal vertex and whose terminal vertex w is a white terminal vertex of Γ_X.*

Proof. The graph Γ_X is a tree that contains no horned trees by theorem 5.4. Let b'' be the black terminal vertex of Γ_X and let v be the white vertex adjacent to b''.

Suppose that Γ_X has no black vertices of degree 3. The graph Γ_X is a linear graph. Orient the graph Γ_X so that vertices are ordered as $b_1 - w_1 - b_2 - ... - b_{r+1} - w_{r+1}$ with corresponding edge labels $n_1 - \ldots - m_{r+1} - n_{r+1}$ where $b_1 = b''$. By assumption the labels $m_r = 1, n_r = 2$. If there exists a subgraph $w_{i-1} - b_i - w_i$ for $1 < i < r + 1$ with successive labels $m_i = 2, n_i = 1$ then Γ_X contains a horned tree. Therefore each subgraph $w_{i-1} - b_i - w_i$ for $1 < i < r+1$ has successive labels $m_i = 1, n_i = 2$. The linear graph L with initial vertex w_1 and terminal vertex w_{r+1} in Γ_X is an O-string. L-prune the subgraph $w_1 - b_2 - ... - b_{r+1} - w_{r+1}$ of Γ_X, the resulting graph Γ' has vertices $b_1 - w_1' - b_2' - w_2'$ with successive edge labels $3, 1, 2^r$. A presentation of the fundamental group of $X_{\Gamma'}$ is given by:

$$\{x_1 : x_1^{3*2^r} = 1\}.$$

Then $\pi_1(X) \cong \mathbb{Z}_{3*2^r}$ where r is the number of edges with label 2 in the O-string L.

□

The dihedral group of order $2n$ will be denoted by D_n.

Lemma 6.5. *Let X be a pruned trivalent 2-stratifold where Γ_X has a label 2 for all edges incident to a terminal white vertex of genus 0. Let Γ_X have all white vertices of genus 0, all terminal vertices are white, and there is exactly one* *white vertex v'' of degree 3 while all other white vertices are of degree ≤ 2. Let e_i be the edges incident to v'' for $1 \leq i \leq 3$. Let L^i be a linear subgraph of Γ_X whose initial vertex is v'', whose terminal vertex w is a terminal vertex of Γ_X, and L^i contains e_i. If $\pi_1(X)$ is finite then all of the following hold:*

1. *The linear subgraph L^i is an O-string.*
2. *There exists an L^i for $i = 1, 2$ of Γ_X that contains only one edge labelled with 2.*
3. *The fundamental group $\pi_1(X)$ is isomorphic to D_{2^k}, where the integer $k > 0$ corresponds to the minimal number of edges with label 2 in all L^3 of Γ_X.*

Proof. The graph Γ_X is a tree that contains neither a horned tree disjoint from v'' nor a horned tree with v'' as a terminal vertex by theorem 5.4.

Suppose that Γ_X has no black vertices of degree 3. Define L_i to be the linear subgraph whose initial vertex is v'', whose terminal vertex is a terminal vertex of Γ_X, and L_i contains the edge e_i. Since $\pi_1(X)$ is finite, each L_i is a p-string of length $2p_i$. A presentation of $\pi_1(X)$ is given by the following:

$$\{c_1, c_2, c_3 : c_1^{2^{p_1}} = 1, c_2^{2^{p_2}} = 1, c_3^{2^{p_3}} = 1, c_1c_2c_3 = 1\}.$$

Then $\pi_1(X)$ is an F-group. Each $p_i > 0$ and so the presentation represents a finite non-cyclic F-group. Therefore without a loss of generality, we have $p_1 = 1$, $p_2 = 1$, and $p_3 \geq 1$. It follows that L_1, L_2 are p-strings of length 2, L_3 is a p-string of length $2p_3$, and $\pi_1(X_\Gamma)$ is the dihedral group $D_{2^{p_3}}$.

Suppose that Γ_X contains $n > 0$ black vertices of degree 3. Let b be a black vertex of degree 3 that is adjacent to the vertices v_1, v_2, v_3 such that v_i is the initial vertex of a terminal linear subgraph T_i for $i = 1, 2$. Since $\pi_1(X)$ is finite, T_i is a p-string. Let the terminal vertex of T_i, which is terminal vertices of Γ_X, be called t_i. Apply operation $B1$ to $st(b) \cup T_1 \cup T_2$. Let the associated p-string be called T' and let the terminal vertex of T' and Γ' be called t'.

The fundamental group $\pi_1(X_\Gamma)$ is isomorphic to $\pi_1(X_{\Gamma'})$ and by the induction hypothesis, $\pi_1(X_{\Gamma'})$ is isomorphic to D_{2^k} for $k > 0$.

Let L' be a linear subgraph of Γ' whose initial vertex is v'' and whose terminal vertex w' is a terminal vertex of Γ'. Then L' is an O-string. If L' is disjoint from $T' \setminus v_3$ then L' is disjoint from T'. Therefore L' is an O-string in Γ_X that is disjoint from $v_3 \cup st(b) \cup T_1 \cup T_2$. Now assume L' is not disjoint from $T' \setminus v_3$. Then the terminal vertex w' of L' is t' which is the terminal vertex of T'. It follows by Lemma 6.1 that every linear subgraph L of Γ_X whose initial vertex is v'' and whose terminal vertex is t_i of Γ_X is an O-string.

By the induction hypothesis, there exists an O-string L'_i that contains e_i with initial vertex is v'', terminal vertex is a terminal vertex of Γ', and exactly p_i edges with label 2 where $p_i = 1$ if $i = 1, 2$ and $p_i \geq 1$ if $i = 3$. If L'_i is disjoint from $T' \setminus v_3$ then L'_i is contained in Γ_X and the result follows. If L'_i is not disjoint from $T' \setminus v_3$ then the terminal vertex w' of L' is t' which is the terminal vertex of T'. By Lemma 6.1 there exists an O-string of Γ_X whose initial vertex is v'' and whose terminal vertex is t_i of Γ_X with exactly p_i edges with label 2.

□

7 Trivalent 2-stratifolds with Finite Fundamental Group

We describe the necessary and sufficient conditions on a trivalent 2-stratifold X for $\pi_1(X_\Gamma)$ to be finite. All X in this section are assumed to be trivalent and satisfy a set of necessary conditions from lemma 2.2. A 2-stratifold X_Γ with a graph Γ that contains a vertex of genus -1 or a black terminal vertex is never 1-connected. For graphs Γ with all white terminal vertices and all white vertices of genus 0, the associated 2-stratifold X_Γ can be 1-connected. **We further assume that X_Γ is not 1-connected and X_Γ is pruned.**

We define core-reduced graphs for X_Γ which are pruned subgraphs of Γ_X that carry the fundamental group information of X_Γ.

A vertex of Γ with degree > 2 will be called a **branch vertex**. Let b_0 be a black branch vertex of distance 1 from a terminal vertex w_0 and let C_1, C_2 be subgraphs of Γ corresponding to the components of $\Gamma \setminus (st(b_0) \cup w_0)$. Then such a black branch vertex b_0 is an called **outermost** if at least one C_i contains no black branch vertices distance 1 to a terminal vertex. We refer to a labelled graph Γ as 1-connected if X_Γ is 1-connected.

If the graph Γ does not contain a black branch vertex of distance 1 to a terminal vertex then Γ is core-reduced. If Γ contains a black branch vertex of distance 1 to a terminal vertex we let $B = \{b_{01}, \ldots, b_{0k}\}$ be the set of all outermost black branch vertices where each b_{0i} has distance 1 from a terminal vertex w_{0i}. Choose a component of $\Gamma \setminus (st(b_{0i}) \cup w_{0i})$ corresponding to a subgraph C_i of Γ that does not contain a black branch vertex of distance 1 to a terminal vertex to be denoted T_{0i}. If there exists at least two components T_{0i} that are not 1-connected let $\Gamma_0 = \emptyset$. If one component T_{0i} is not 1-connected and $\Gamma \setminus (T_{0i} \cup st(b_{0i}) \cup w_{0i})$ is not 1-connected then let $\Gamma_0 = \emptyset$. If each T_{0i} is 1-connected and $\Gamma \setminus (T_{0i} \cup st(b_{0i}) \cup w_{0i})$ is not 1-connected then let $\Gamma_0' = \Gamma \setminus (\bigcup st(b_{0i}) \cup \bigcup w_{0i} \cup \bigcup T_{0i})$. If exactly one component T_{0i} is not 1-connected and $\Gamma \setminus (T_{0i} \cup (st(b_{0i}) \cup w_{0i})$ is 1-connected then let $\Gamma_0' = T_{0i}$. If Γ_0' is pruned then let $\Gamma_0 = \Gamma_0'$, otherwise let Γ_0 be the pruned Γ_0'. For $\Gamma_0 \neq \emptyset$, we have that $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_0})$ since $r^{-1}(b_{i0})$ is contractible in X_Γ. For $\Gamma_0 = \emptyset$, we have that $\pi_1(X_\Gamma)$ is infinite.

By induction, If Γ_{n-1} contains a black branch vertex of distance 1 to a terminal vertex we let $B_{n-1} = \{b_{n-1,1}, \ldots, b_{n-1,k_{n-1}}\}$ be the set of all outermost black branch vertices where each $b_{n-1,i}$ has distance 1 from a terminal vertex $w_{n-1,i}$. Choose a component of $\Gamma_{n-1} \setminus (st(b_{n-1,i}) \cup w_{n-1,i})$ corresponding to a subgraph C_i of Γ_{n-1} that does not contain a black branch vertex of distance 1 to a terminal vertex to be denoted $T_{n-1,i}$. If there exists at least two components $T_{n-1,i}$ that are not 1-connected let $\Gamma_n = \emptyset$. If one component $T_{n-1,i}$ is not 1-connected and $\Gamma \setminus (T_{n-1,i} \cup st(b_{n-1,i}) \cup w_{n-1,i})$ is not 1-connected then let $\Gamma_n = \emptyset$. If each $T_{n-1,i}$ is 1-connected and $\Gamma \setminus (T_{n-1,i} \cup st(b_{n-1,i}) \cup w_{n-1,i})$ is not 1-connected then let $\Gamma_n' = \Gamma_{n-1} \setminus (\bigcup st(b_{n-1,i}) \cup \bigcup w_{n-1,i} \cup \bigcup T_{n-1,i})$. If exactly one component $T_{n-1,i}$ is not 1-connected and $\Gamma_{n-1} \setminus (T_{n-1,i} \cup st(b_{n-1,i}) \cup w_{n-1,i})$ is 1-connected then let $\Gamma_n' = T_{n-1,i}$. If Γ_n' is pruned the let $\Gamma_n = \Gamma_n'$, otherwise let Γ_n be the pruned Γ_n'.

We define our **core reduced graph** Γ_C of Γ as follows:

$$\Gamma_C = \begin{cases} \emptyset, & \text{if } \Gamma_n = \emptyset \text{ for some } n \geq 0, \text{otherwise} \\ \Gamma_n, & \text{for the smallest } n \text{ such that } \Gamma_n \text{ does not contain a black branch vertex of} \\ & \text{distance 1 to a terminal vertex} \end{cases}$$

For a core reduced graph Γ_C of Γ where $\Gamma_C \neq \emptyset$, we have that $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$. While if $\Gamma_C = \emptyset$ then $\pi_1(X_\Gamma)$ is infinite.

A **pseudo-projective plane of order** $k > 2$ is a 2-stratifold that is obtained by attaching a 2-cell to a circle by the map $z \to z^k$. A pseudo-projective plane of order 3 is a trivalent 2-stratifold. A closed 2-manifold is considered to be a trivalent 2-stratifold.

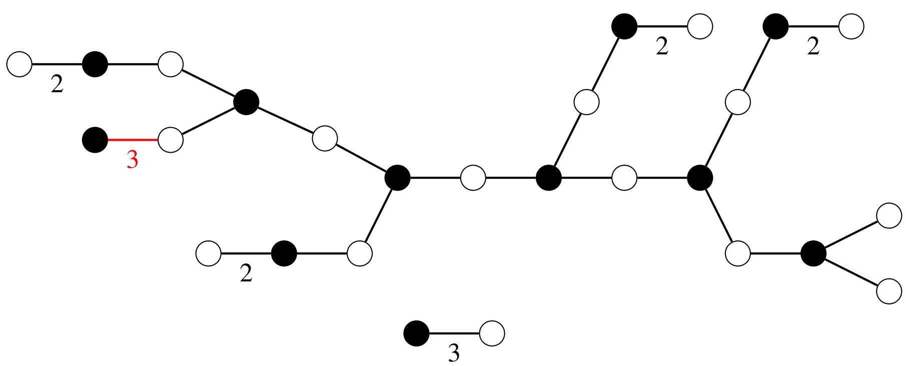

Figure 5: A trivalent graph Γ and its core reduced graph Γ_C.

Lemma 7.1. *Let Γ be a bicolored pruned trivalent graph such that X_Γ is a trivalent 2-stratifold that has finite (nontrivial) fundamental group. Let Γ_C be the core reduced graph of Γ. Then Γ is one of the cases below:*

1. *The graph Γ has exactly one black terminal vertex and all white vertices are genus 0. Then the graph Γ_C contains exactly one black terminal vertex, all white vertices are genus 0, and either all edges of Γ_C incident to a terminal white vertex have label 2 or X_{Γ_C} is a pseudo-projective plane of order 3.*

2. *The graph Γ has exactly one white vertex of genus -1 while all other white vertices are genus 0 and all terminal vertices are white. Then the graph Γ_C either contains one white vertex of genus -1 while all other white vertices are genus 0, all terminal vertices are white, and all edges of Γ_C incident to a terminal white vertex of genus 0 have label 2 or X_{Γ_C} is a projective plane.*

3. *The graph* Γ *has all white terminal vertices and white vertices are of genus* 0*. Then the graph* Γ_C *contains all white vertices of genus* 0*, all terminal vertices are white, and all edges of* Γ_C *incident to a terminal vertex have label* 2*.*

Proof. The graph Γ_C is a pruned subgraph of Γ. Since $\pi(X_\Gamma)$ is finite, $\Gamma_C \neq \emptyset$.

(1.) The graph Γ_C contains at most one black terminal vertex and all white vertices are of genus 0. Suppose that Γ_C does not contain a black terminal vertex. If Γ is not 1-connected then Γ_C is not 1-connected. Let Γ_0 be the subgraph of Γ corresponding to Γ_C. Attach to each black vertex that is not the terminal black vertex and is not contained in the subgraph Γ_0 of Γ a white vertex of genus 0 with edge label 1. Then there is an epimorphism from $\pi_1(X_\Gamma) \to \mathbb{Z}_3 \star \pi_1(X_{\Gamma_C})$. The graph Γ_C contains a black terminal vertex.

The graph Γ_C contains no terminal q-strings and no black branch vertex of distance 1 to a terminal vertex. Let v be a white terminal vertex of Γ_C. If v is not contained in a terminal p-string then v is adjacent to the black terminal vertex and X_{Γ_C} is a pseudo-projective plane of order 3. Otherwise v is contained in a terminal p-string and the edge label incident to v is 2.

(2.) The graph Γ_C contains at most one white vertex of genus -1 while all other vertices are genus 0 and all terminal vertices are white. Suppose that Γ_C does not contain a white vertex of genus -1. If Γ is not 1-connected then Γ_C is not 1-connected. Let Γ_0 be the subgraph of Γ corresponding to Γ_C. Attach to each black vertex not contained in the subgraph Γ_0 of Γ a white vertex of genus 0 with edge label 1. Then there is an epimorphism from $\pi_1(X_\Gamma) \to \mathbb{Z}_2 \star \pi_1(X_{\Gamma_C})$. The graph Γ_C contains the white vertex of genus -1.

The graph Γ_C contains no terminal q-strings and no black branch vertex of distance 1 to a terminal vertex. If Γ_C contains a white terminal vertex v of genus 0 then v is contained in a terminal p-string and the edge label incident to v is 2. If Γ_C contains no white terminal vertices of genus 0 then X_{Γ_C} is a projective plane.

(3.) The graph Γ_C contains all white terminal vertices and all white vertices are of genus 0. The graph Γ_C contains no terminal q-strings and no black branch vertex of distance 1 to a terminal vertex. If v is a white terminal vertex of genus 0 then the incident edge label is 2.

□

We determine the finite trivalent 2-stratifold groups.

Theorem 7.2. *Let* Γ *be a bicolored pruned trivalent graph. If* X_Γ *has finite fundamental group then* $\pi_1(X_\Gamma)$ *is isomorphic to either* $\mathbb{Z}_{2^{k+1}}$*,* $\mathbb{Z}_{3*2^k}$*,* $D_{2^{k+1}}$ *where* $k \geq 0$*.*

Proof. Let Γ_C be the core reduced graph of Γ.

Suppose that Γ has exactly one black terminal vertex and all white vertices are genus 0. By lemma 7.1, the graph Γ_C contains exactly one black terminal vertex, all white vertices are genus 0, and either all edges of Γ_C incident to a terminal white vertex have label 2 or X_{Γ_C} is a pseudo-projective plane of order 3. If X_{Γ_C} is a pseudo-projective plane of order 3 then $\pi_1(X_\Gamma) \cong \mathbb{Z}_3$. Otherwise by theorem 5.4, Γ_C has all white vertices of degree ≤ 2, and contains no horned tree. Let L be a linear subgraph of Γ_C whose initial vertex v is the white vertex adjacent to the black terminal vertex and whose terminal vertex w is a white terminal vertex of Γ_C. Then by lemma 6.4, L is an O-string, $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_{3*2^k}$ where $k > 0$, and the integer k corresponds to the minimal number of edges with label 2 in all L whose initial vertex v is the white vertex adjacent to the black terminal vertex and whose terminal vertex w is a white terminal vertex of Γ_C.

Suppose that Γ has exactly one white vertex of genus -1 while all other white vertices are genus 0 and all terminal vertices are white. By lemma 7.1, the graph Γ_C either contains one white vertex of genus -1 while all other white vertices are genus 0, all terminal vertices are white, and all edges of Γ_C incident to a terminal white vertex of genus 0 have label 2 or X_{Γ_C} is a projective plane. If X_{Γ_C} is a projective plane then $\pi_1(X_\Gamma) \cong \mathbb{Z}_2$. Otherwise by theorem 5.4, the white vertex of genus -1 of Γ_C is terminal and has incident edge label 1, Γ_C contains all white vertices of degree ≤ 2, and Γ_C contains no horned tree. Let L be a linear subgraph of Γ_C whose initial vertex v is the white vertex of genus -1 and whose terminal vertex w is a white terminal vertex of Γ_C where $w \neq v$. Then by lemma 6.3, L is an O-string, $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_{2^k}$ where $k > 1$, and the integer k corresponds to the minimal number of edges with label 2 in all L whose initial vertex v is the white vertex of genus -1 and whose terminal vertex w is a white terminal vertex of Γ_C.

Suppose that Γ contains all white vertices of genus 0 and all terminal vertices are white. By lemma 7.1, Γ_C contains all white vertices of genus 0, all terminal vertices are white and all edges of Γ_C incident to a terminal white vertex has label 2. By theorem 5.4, either Γ_C has all white vertices of degree ≤ 2 and contains at most one horned tree or Γ_C has exactly one white vertex v'' of degree 3 while all other white vertices are of degree ≤ 2 and contains no horned tree H_T such that either v'' and H_T are disjoint or v'' is a terminal vertex of H_T. We now look at these two cases.

Suppose that Γ_C has all white vertices of degree ≤ 2 and contains at most one horned tree. By lemma 6.2, Γ_C contains a horned tree H_T and if L is a linear subgraph of Γ_C whose initial vertex v is a terminal vertex of H_T and whose terminal vertex w is a white terminal vertex of Γ_C where $L \cap H_T = v$ and $w \neq v$ then L is an O-string. Further by lemma 6.2, $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_{2^{k+1}}$ where the integer $k = 0$ if H_T contains a terminal vertex of Γ_X and $k > 0$ otherwise. The integer $k > 0$ corresponds to the minimal number of edges with label 2 in all linear subgraphs L whose initial vertex v is a terminal vertex of H_T and whose terminal vertex w is a terminal vertex of Γ_X where $L \cap H_T = v$ and $w \neq v$.

Suppose that Γ_C has exactly one white vertex v'' of degree 3 while all other white vertices are of degree < 3, and contains no horned tree H_T such that either v'' and H_T are disjoint or v'' is a terminal vertex of H_T. Let e_i be the edges incident to v'' for $1 \leq i \leq 3$. Let L^i be a linear subgraph of Γ_X whose initial vertex is v'', whose terminal vertex w is a terminal vertex of Γ_X, and L^i contains e_i. By lemma 6.5, the linear subgraph L^i is an O-string, there exists an L^i for $i = 1, 2$ of Γ_X that contains only one edge labelled with 2, and the fundamental group $\pi_1(X)$ is isomorphic to D_{2^k}, where the integer $k > 0$ corresponds to the minimal number of edges with label 2 in all L^3 of Γ_X.

□

We now state our main classification results.

Theorem 7.3. *Let Γ be a bicolored pruned trivalent graph. Then $\pi_1(X_\Gamma) \cong \mathbb{Z}_3$ if and only if the following hold:*

1. *The graph Γ is a tree that has exactly one black terminal vertex, all white vertices are genus 0;*
2. *The core reduced graph $\Gamma_C \neq \emptyset$, Γ_C is the core reduced graph of Fig. 4.3, and X_{Γ_C} is a pseudo-projective plane of order 3.*

Proof. Suppose $\pi_1(X_\Gamma) \cong \mathbb{Z}_3$. Since $\pi_1(X_\Gamma)$ is finite the result follows from the proof of theorem 7.2. Suppose that condition 1. and 2. holds. Then $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_3$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$.

□

Theorem 7.4. *Let Γ be a bicolored pruned trivalent graph. Then $\pi_1(X_\Gamma) \cong \mathbb{Z}_{3*2^k}$ for $k > 0$ if and only if the following hold:*

1. *The graph Γ is a tree that has exactly one black terminal vertex and all white vertices are genus 0;*
2. *The core reduced graph $\Gamma_C \neq \emptyset$ and all edges of Γ_C incident to a terminal white vertex of genus 0 have label 2;*
3. *The graph Γ_C contains exactly one black terminal vertex, all white vertices are genus 0 and have degree ≤ 2, and the graph Γ_C contains no horned trees;*
4. *Let L be an linear subgraph of Γ_C whose initial vertex v is the white vertex adjacent to the black terminal vertex and whose terminal vertex w is a white terminal vertex of Γ_C. Then L is an O-string that contains $r \geq k$ edges with label 2 and there exists at least one L that contains k edges with label 2.*

Proof. Suppose $\pi(X_\Gamma) \cong \mathbb{Z}_{3*2^k}$ for $k > 0$. Since $\pi_1(X_\Gamma)$ is finite the result follows from the proof of theorem 7.2.

Suppose that conditions 1. thru 4. holds. By the proof of lemma 6.4, $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_{3*2^k}$ for $k > 0$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$.

□

Theorem 7.5. *Let Γ be a bicolored pruned trivalent graph. Then $\pi_1(X_\Gamma) \cong \mathbb{Z}_2$ for if and only if either 1.(a)-1.(b) or 2.(a)-2.(e) are satisfied.*

1. (a) *The graph Γ has exactly one white vertex of genus -1 while all other white vertices are genus 0 and all terminal vertices are white;*
   (b) *The core reduced graph $\Gamma_C \neq \emptyset$, Γ_C is a single white vertex of genus -1 with no edges, and X_{Γ_C} is a projective plane;*
2. (a) *The graph Γ contains all white vertices of genus 0 and all terminal vertices are white*

(b) *The core reduced graph $\Gamma_C \neq \emptyset$ and all edges of Γ_C incident to a terminal vertex of genus 0 have label 2;*
(c) *The core reduced Γ_C contains all white vertices of genus 0 and all white vertices are of degree ≤ 2, all terminal vertices are white, and Γ_C contains a horned tree H_T.*
(d) *If L is a linear subgraph of Γ_C whose initial vertex v is a terminal vertex of H_T and whose terminal vertex w is a terminal vertex of Γ_C where $L \cap H_T = v$ and $w \neq v$ then L is an O-string.*
(e) *The horned tree H_T contains a terminal vertex of Γ_C*

Proof. Suppose $\pi(X_\Gamma) \cong \mathbb{Z}_2$. Since $\pi_1(X_\Gamma)$ is finite the result follows from the proof of theorem 7.2.

Suppose that conditions 2.(a)-2.(e) holds. Then by the proof of lemma 6.2, $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_2$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$.

Suppose that condition 1.(a)-1.(b) holds. Then $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_2$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$.

□

Theorem 7.6. *Let Γ be a bicolored pruned trivalent graph. Then $\pi_1(X_\Gamma) \cong \mathbb{Z}_{2^{k+1}}$ for $k > 0$ if and only if either 1.(a)-(d) or 2.(a)-(d) are satisfied.*

1. (a) *The graph Γ has exactly one white vertex of genus -1 while all other white vertices are genus 0 and all terminal vertices are white*
   (b) *The core reduced graph $\Gamma_C \neq \emptyset$ and all edges of Γ_C incident to a terminal vertex of genus 0 have label 2;*
   (c) *The core subgraph Γ_C has exactly one white terminal vertex of genus -1 with incident edge label 1 while all other white vertices are genus 0, all white vertices are of degree ≤ 2 and all terminal vertices are white, and Γ_C contains no horned trees.*
   (d) *Let L be a linear subgraph of Γ_C whose initial vertex v is the white vertex of genus -1 and whose terminal vertex w is a terminal vertex of Γ_C where $w \neq v$. Then L is an O-string that contains $r \geq k$ edges with label 2 and there exists at least one L that contains k edges with label 2.*
2. (a) *The graph Γ contains all white vertices of genus 0 and all terminal vertices are white*
   (b) *The core reduced graph $\Gamma_C \neq \emptyset$ and all edges of Γ_C incident to a terminal vertex of genus 0 have label 2;*
   (c) *The core reduced graph Γ_C contains all white vertices of genus 0 and are of degree ≤ 2, all terminal vertices are white, and Γ_C contains a horned tree H_T.*
   (d) *Let L be a linear subgraph of Γ_C whose initial vertex v is a terminal vertex of H_T and whose terminal vertex w is a terminal vertex of Γ_C where $L \cap H_T = v$ and $w \neq v$. Then L is an O-string that contains $r \geq k$ edges with label 2 and there exists at least one L that contains k edges with label 2.*

Proof. Suppose $\pi(X_\Gamma) \cong \mathbb{Z}_{2^{k+1}}$. Since $\pi_1(X_\Gamma)$ is finite the result follows from the proof of theorem 7.2.

Suppose that either conditions 1.(a)-1.(d) or 2.(a)-2.(d) holds. Then by the proof of lemma 6.3 or lemma 6.2 respectively, $\pi_1(X_{\Gamma_C}) \cong \mathbb{Z}_{2^{k+1}}$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$.

□

Theorem 7.7. *Let Γ be a bicolored pruned trivalent graph. Then $\pi_1(X_\Gamma) \cong D_{2^{k+1}}$ for $k \geq 0$ if and only if the following hold:*

1. *The graph Γ is a tree that has all white terminal vertices and white vertices are of genus 0*
2. *The core reduced graph $\Gamma_C \neq \emptyset$ and all edges of Γ_C incident to a terminal white vertex of genus 0 have label 2;*
3. *The core reduced graph Γ_C has all white vertices of genus 0 and all terminal vertices are white, there is exactly one white vertex v'' of degree 3 while all other white vertices are of degree ≤ 2, and Γ_C contains no horned tree H_T such that either v'' and H_T are disjoint or v'' is a terminal vertex of H_T*
4. *Let L^i be a linear subgraph of Γ_C whose initial vertex is v'', whose terminal vertex w is a terminal vertex of Γ_C, and L^i contains e_i. The linear subgraph L^i is an O-string, there exists an L^i for $i = 1, 2$ of Γ_C that contains only one edge labelled with 2, and all L^3 contains $r \geq k$ edges with label 2 and there exists at least one L^3 that contains k edges with label 2.*

Proof. Suppose $\pi(X_\Gamma) \cong D_{2^{k+1}}$ for $k > 0$. Since $\pi_1(X_\Gamma)$ is finite the result follows from the proof of theorem 7.2. Suppose that either conditions 1-4 holds. Then by the proof of lemma 6.5, $\pi_1(X_{\Gamma_C}) \cong D_{2^{k+1}}$ and $\pi_1(X_\Gamma) \cong \pi_1(X_{\Gamma_C})$. □

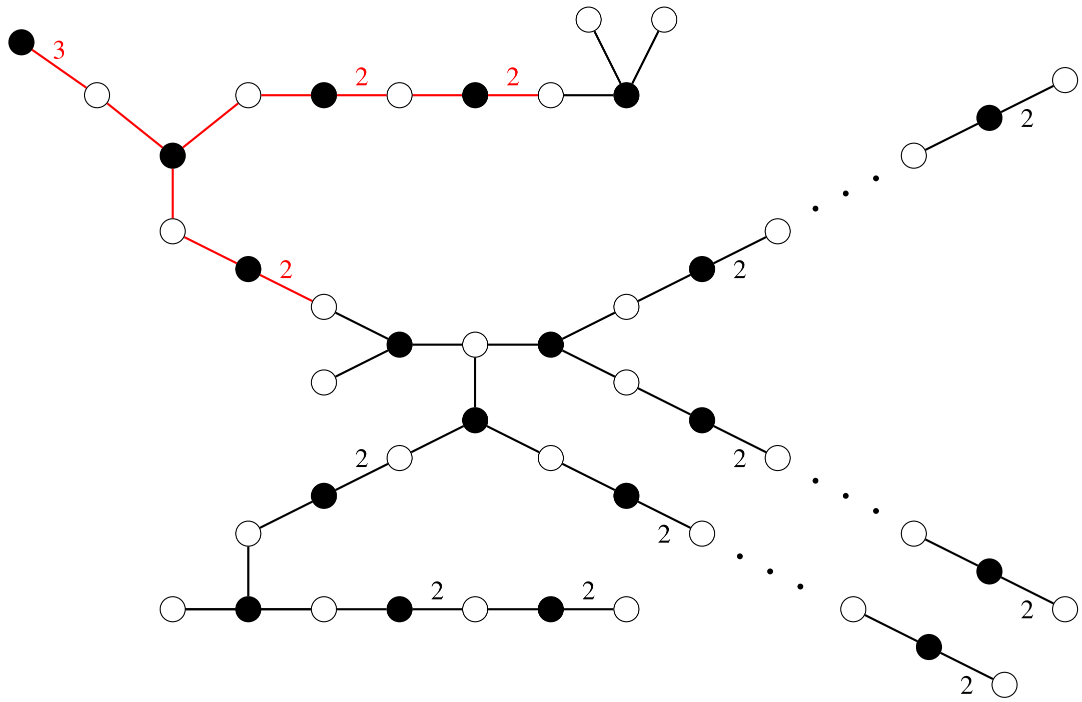

Figure 6: A trivalent graph Γ and its core reduced graph Γ_C that satisfies the conditions of Theorem 7.4.

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